

Minimum Viable Scale

Bellman Barriers and the Economics of Survival*

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Abstract

With increasing returns, complementary inputs and bounded labor, physical maintenance need not make preservation optimal. We characterize the complete sustainable set. In a patient constant elasticity of substitution (CES) family at sufficiently high productivity, every optimum below a unique welfare frontier becomes extinct; every optimum above it survives. Both regimes attain equal welfare at the frontier, with separated choices of next period's capital and labor and an upward welfare kink. Yet extinct histories can approximate optimal welfare where every optimum survives. We develop First Passage Bellman Barriers to verify entry and permanence for every optimum from feasible plans and independent welfare bounds. We construct barriers from calibrated block losses and, under compactness and attainment, characterize certificate existence and suprenal margins. In CES, feasible maintenance and favorable attainable returns coexist with optimal extinction. In renewal, state dependent barriers succeed where constant barriers fail; in its periodic subcase, every calibrated constant reference test shorter than the wear cycle fails. Neither application requires the full value function or an optimal witness.

Keywords. minimum viable scale; optimal survival; increasing returns; Bellman barriers; dynamic verification.

JEL codes. C61, D24, D61, O41.

1 Introduction

Increasing returns make absolute scale matter. Expanding capital and labor together raises output more than proportionally, so a small productive system is not simply a scaled down large one. Reaching a more productive scale can justify saving and work that would not pay at a smaller scale. These gains can favor sharply separated choices between building a productive base and consuming its capital down; they do not by themselves force corner solutions.

Total available labor is normalized to one. Labor is complementary to capital, so the inputs cannot expand together indefinitely. Work sacrifices leisure, and saving sacrifices consumption. As labor approaches one, leisure utility tends to minus infinity and the marginal utility cost of work diverges; zero consumption also gives utility minus infinity. Full labor and zero consumption are therefore excluded from optima with finite welfare. These restrictions do not make production

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concave. Positive consumption and leisure remain compatible with both preservation and depletion; zero labor is not generally excluded.

The physical maintenance limits follow from the technology. With little capital, even full labor produces too little to replace depreciation: scarcity need not bring a high return. With abundant capital, bounded labor limits output while replacement needs keep rising. Depending on productivity, the sustainable set is empty, a singleton, or an interval. At its endpoints, constant capital requires full labor and zero consumption; the interval's interior permits positive consumption and leisure. Its upper boundary limits constant maintenance, not survival after drawing down a larger stock.

Physical sustainability compares output with depreciation. The Euler condition compares marginal sacrifices and returns across dates. Optimal preservation compares the lifetime welfare of complete histories. The paper's attainable return test uses a feasible labor choice, not necessarily an optimal one; even a necessary Euler condition does not establish global optimality here. The analytical example below has stocks that pass both the maintenance and attainable return tests, yet every optimum becomes extinct.

Our economic contribution characterizes the gap between maintenance and optimal preservation. In the patient constant elasticity of substitution (CES) family at sufficiently high productivity, maintenance is feasible on $[\kappa_p, K_p^+]$, but a unique welfare frontier S_p lies strictly inside that interval. Below S_p , every optimum becomes extinct; above it, every optimum survives. At S_p , both regimes attain the same welfare. Thus at $\kappa_p < K < S_p$, preserving capital with positive consumption and leisure is feasible but not worthwhile. Figure 1 places the physical boundaries and the welfare frontier on the same capital axis.

Our methodological contribution, First Passage Bellman Barriers (FPBB), verifies entry and permanence from feasible plans and independent welfare bounds, without the full value function. It supplies the productive anchor used to establish the welfare frontier.

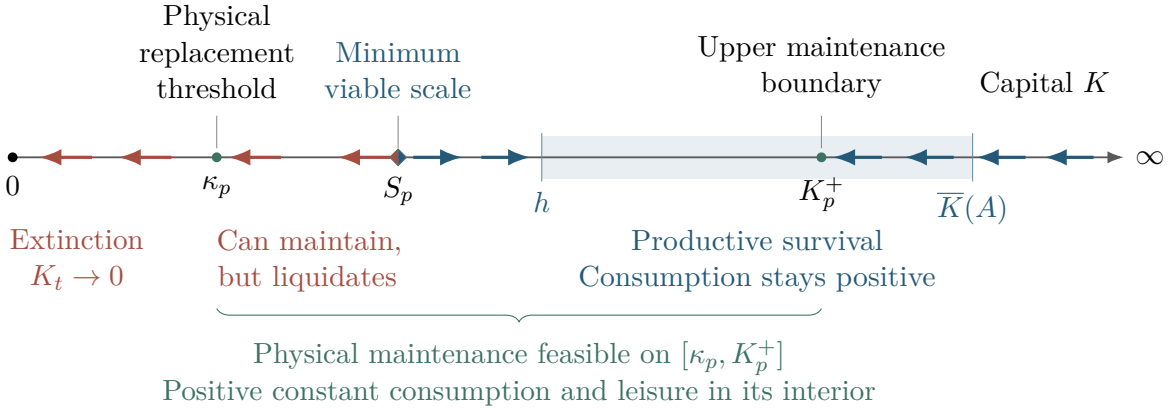


Figure 1: Optimal capital movements and physical maintenance in the patient CES family at sufficiently high productivity. Small arrows show whether optimal capital rises or falls in the next period; a choice may jump across marked stocks. Below S_p every optimum declines to zero. The split marker at S_p indicates optimal choices in both directions. Above S_p every optimum enters $[h, \bar{K}(A)]$ and remains there, with consumption bounded away from zero. The shaded region represents productive survival, without asserting convergence to a single steady state. Direction inside $[h, K_p^+]$ is left unspecified. At either physical endpoint, constant capital requires full labor and zero consumption. Proposition 1, Theorem 1 and Lemma 4 establish the indicated movements. Positions and distances are schematic.

Preservation must repay both its transition and operating costs in consumption and leisure. Reaching a more productive range takes saving; operating there takes work. Drawing down capital releases resources for consumption instead. At the frontier these alternatives yield equal lifetime welfare, although their eventual outcomes differ. The two regimes need not contain just two optimal histories. Extinction means capital tends to zero, not that production immediately stops. Productive survival means capital eventually remains in a bounded interval away from zero, not that it converges to a single steady state. Under the classification hypotheses, survival also occurs from stocks above K_p^+ , after an initial decline.

The change in eventual behavior also changes current choices. At the frontier, next period's capital and labor have strictly separated limits: productive optima carry more capital forward and work more than extinct optima. We refer to this successor capital choice as investment. An additional unit of capital is more valuable on the productive continuation, so the marginal value of capital jumps upward. A small capital transfer between identical economies initially at the frontier can raise combined welfare: the recipient survives and the donor becomes extinct. This allocation gain follows from the ordering of marginal welfare across the two regimes, rather than from physical infeasibility at the donor's stock. The gain survives sufficiently small proportional transfer losses. The same strict slope separation makes the frontier locally Lipschitz in all seven primitives: its local movement is bounded by a constant times the change in fundamentals.

Following an optimum for many dates and then liquidating produces extinct histories arbitrarily close to optimal welfare, even where every optimum survives. Initial welfare accuracy cannot settle the behavioral question. FPBB instead certifies finite entry into a prescribed target and permanent residence there by every optimizer, using welfare bounds that need not identify either value or an optimal policy.

A certificate separates a staging region from the target and excludes two ways of failing to reach and remain there. A regional Bellman barrier bounds the welfare of every history that waits in staging forever; a specified feasible plan reaches the target, remains there and earns more. An adverse departure leaves both regions before entry, or leaves the target afterward. At the first such decision, a feasible continuation with the required behavior earns more than the departing action's current payoff plus a discounted upper bound on all subsequent welfare, including recovery. Replacing the adverse action and its continuation with this feasible plan preserves the past and raises welfare. The two comparisons establish finite entry and permanence.

The general theory constructs barriers and identifies their limits. Accumulated losses over finite blocks yield a bounded regional barrier, with a sharp margin calibrated to feasible welfare. Immediate loss may vanish. Under compactness and attainment, optimal selection is equivalent to strict certificate existence; losses from perpetual waiting and adverse departure determine the supremum of uniform margins. The converse uses exact value to characterize certificates; the applications construct them from independent inputs.

The same patient CES specialization also violates global concavity and increasing differences in the period payoff after maximizing over labor. These violations occur at stocks different from those where favorable maintenance and return tests coexist with optimal extinction. Regional verification and local order nevertheless classify every optimum. The productive anchor proof supplies an explicit barrier, feasible witness and independent continuation ceiling.

A separate renewal economy shows why longer blocks can succeed where calibrated constant reference tests fail. No constant regional barrier can both bound perpetual waiting and satisfy the required weak comparison with feasible renewal welfare. In its periodic subcase, every calibrated constant reference test shorter than the wear cycle fails, whereas the full cycle construction certifies

renewal and permanence. The renewal certificate can use a strictly suboptimal witness and need not match the continuation bound.

Critical stocks and optimal extinction are established subjects in nonconcave growth (Skiba, 1978; Majumdar and Mitra, 1982; Dechert and Nishimura, 1983). Akao, Kamihigashi, and Nishimura (2011) distinguish sustainable from critical capital, and Kamihigashi and Roy (2007) obtain global results without concave production. We classify optimal histories with endogenous labor, establish strict separation of successor capital, labor and marginal welfare at the frontier, and relate optimal preservation to physical maintenance. Envelope and indifference principles are established (Milgrom and Segal, 2002; Wagener, 2020); ordering marginal welfare across optimal histories determines the direction of the kink and why a local capital transfer raises aggregate welfare.

FPBB builds on Bellman verification and first exit replacement (Bellman, 1957; Stokey and Lucas, 1989; Akao, Ishii, Kamihigashi, and Nishimura, 2022). First passage analysis in physics and barrier option pricing motivated the approach (Redner, 2001; Merton, 1973); dual option valuation combines feasible lower values with upper bounds (Rogers, 2002; Haugh and Kogan, 2004). Global dual verification, finite step Lyapunov conversion and discounted storage theory provide close antecedents (Gaitsgory, Parkinson, and Shvartsman, 2019; Geiselhart, Gielen, Lazar, and Wirth, 2014; Grüne, Müller, Kellett, and Weller, 2021). The additional results concern which feasible histories maximize welfare. Independent bounds suffice to verify entry and permanence on the complete controlled graph. Calibration yields a sharp constructive margin; a compact converse characterizes strict certificates and their supremal margins. Section 4.8 separates these claims from the established verification ingredients.

2 The economy and physical sustainability

Time is discrete and the horizon is infinite, $t = 0, 1, 2, \dots$. Population is one. Total available labor is normalized to one. The planner chooses labor $L \in [0, 1]$, leaving $1 - L$ as leisure. Given current capital K , the planner chooses consumption C , labor and next period's capital K' . More work sacrifices leisure; more investment sacrifices consumption. Output and retained capital are the same good, so the stock can be consumed rather than preserved. Minimum viable scale is the initial capital threshold for optimal preservation that we will characterize.

The resource constraint, technology, and period utility are

$$\begin{aligned} F_p(K, L) &= A\{wK^\rho + (1 - w)L^\rho\}^{\theta/\rho}, \\ C + K' &\leq qK + F_p(K, L), \quad C, K' \geq 0, \quad 0 \leq L \leq 1, \\ u_p(C, L) &= \eta \log C + (1 - \eta) \log(1 - L), \quad q = 1 - \delta. \end{aligned} \tag{2.1}$$

The primitive vector is $p = (A, \beta, q, \eta, \theta, \rho, w)$, where $A > 0$, $0 < \beta, q, \eta, w < 1$, $\rho < 0$, and $1 < \theta < 2$. Productivity A scales output; β discounts future utility; q is the retained fraction of capital; and η is the consumption weight in utility. The parameter w weights capital in the CES aggregate, ρ governs substitution with elasticity $1/(1 - \rho) < 1$, and θ measures returns when both inputs increase together. Primitive subscripts record dependence on the economy; calculations for a fixed economy may suppress them. Production is defined to be zero on either input axis, its continuous limit. We set $\log 0 = -\infty$. Thus full labor is physically available but gives utility $-\infty$; every optimum with finite welfare has positive consumption and leisure. This preference restriction does not exclude zero labor. There is no additional labor cap and no imposed upper bound on capital.

The restriction $\rho < 0$ makes capital and labor complementary, and both inputs are essential. Joint scaling gives $F_p(aK, aL) = a^\theta F_p(K, L)$ for $a > 0$: expanding both inputs raises output more than proportionally. This is a statement about joint scale, not increasing marginal returns to capital at every stock. The fixed supply of available labor prevents it from expanding indefinitely with capital.

At fixed positive labor, as capital tends to zero, output is asymptotic to $Aw^{\theta/\rho}K^\theta$ and its capital derivative to $\theta Aw^{\theta/\rho}K^{\theta-1}$. Since $\theta > 1$, both output per unit of capital and its marginal product tend to zero. At the other extreme, increasing capital with labor bounded by one cannot raise output above $A(1-w)^{\theta/\rho}$. Depreciation δK is proportional to capital at both extremes. These properties produce the two physical maintenance boundaries when productivity is high enough. The limit on available labor creates the upper physical boundary; the singular leisure payoff separately excludes full labor from an optimum with finite welfare. Neither physical boundary incorporates the consumption and leisure costs of preservation; those enter the planner's welfare comparison.

From initial capital K , a feasible history π specifies consumption, labor and next period's capital at every date, subject to the resource constraint. The planner maximizes $J_K(\pi) = \sum_{t \geq 0} \beta^t u_p(C_t, L_t)$. Write $V_p(K)$ for the value and $G_p(K)$ for the set of next period capital stocks chosen by optimal histories. A history is *extinct* if $K_t \rightarrow 0$ and *productive* if capital eventually remains in a bounded interval bounded away from zero. This definition does not require convergence to a steady state; extinction need not mean immediate shutdown. Since $F_p(K, L) \leq A(1-w)^{\theta/\rho}$, retention $q < 1$ bounds the resources available along every history from a fixed initial stock.

Define

$$\delta = 1 - q, \quad \nu = \frac{1 - \eta}{\eta}, \quad D = \beta^{-1} - q, \quad E = D - \theta\delta. \quad (2.2)$$

Here $D = \beta^{-1} - q$ is the marginal capital product required by an interior stationary consumption Euler equation. Along an interior candidate history, the Euler condition compares the current marginal utility cost of saving with its discounted marginal utility return next period. With constant consumption, it reduces to the stationary return benchmark. The attainable return test below asks whether some feasible labor choice can exceed that benchmark; it does not impose the planner's optimal labor choice or compare lifetime welfare. Nor is a necessary Euler condition sufficient for global optimality in this nonconcave economy. The principal classification concerns the patient region

$$E < 0, \quad \text{equivalently} \quad \beta > [1 + (\theta - 1)(1 - q)]^{-1}. \quad (2.3)$$

Equivalently, $\beta[q + \theta\delta] > 1$. This condition bounds impatience relative to depreciation and increasing returns. Optimal classification also requires sufficiently high productivity; patience alone does not determine whether a given stock survives. Physical sustainability does not require the condition.

2.1 The complete sustainable set

Physical maintenance asks whether output can replace depreciation. Whether maintenance is worthwhile also depends on the consumption and leisure it costs. Define

$$\mathcal{M}_p = \{K > 0 : F_p(K, 1) \geq \delta K\}.$$

This is the set of stocks that can be held constant with nonnegative consumption. It uses the largest physically available labor input. Positive consumption and leisure require its interior, when that interior exists.

Output per unit of capital is low at both ends of the stock range. Complementarity limits production at low capital. At high capital, labor is bounded while replacement needs grow with the stock. The sustainable set can therefore have both a lower and an upper boundary.

For $L > 0$, write $F_p(K, L) = AL^\theta y(K/L)$, where $y(k) = (wk^\rho + 1 - w)^{\theta/\rho}$. Here $k = K/L$ is the capital–labor ratio; at full labor it equals the capital stock. Put

$$\begin{aligned} k_a &= \left[\frac{1-w}{w(\theta-1)} \right]^{1/\rho}, & H_a &= \frac{y(k_a)}{k_a}, \\ k_\Phi &= \left[\frac{(1-\rho)(1-w)}{w(\theta-1)} \right]^{1/\rho}, & \Phi_a &= y'(k_\Phi), \\ A_{\text{rep}} &= \delta/H_a, & A_{\text{ret}} &= D/\Phi_a. \end{aligned} \tag{2.4}$$

The average product peak H_a determines physical maintenance; the marginal product peak Φ_a determines attainable investment returns. Neither determines whether preservation maximizes welfare.

Proposition 1 (Physical sustainability and attainable returns). *The sustainable set is*

$$\mathcal{M}_p = \begin{cases} \emptyset, & A < A_{\text{rep}}, \\ \{k_a\}, & A = A_{\text{rep}}, \\ [\kappa_p, K_p^+], & A > A_{\text{rep}}, \end{cases} \tag{2.5}$$

where $\kappa_p < k_a < K_p^+$ are the two roots of $F_p(K, 1) = \delta K$. Positive constant consumption and leisure are feasible precisely at $K \in (\kappa_p, K_p^+)$ when $A > A_{\text{rep}}$. If $A < A_{\text{rep}}$, every feasible history obeys

$$K_t \leq (q + AH_a)^t K_0 \longrightarrow 0. \tag{2.6}$$

There exist positive inputs with $L < 1$ and $\beta[q + F_{p,K}(K, L)] > 1$ if and only if $A > A_{\text{ret}}$. The labor requirements for strict physical maintenance and for a strict favorable return have lower boundaries

$$\lambda_{\text{rep}} = \left(\frac{\delta}{AH_a} \right)^{1/(\theta-1)}, \quad \lambda_{\text{ret}} = \left(\frac{D}{A\Phi_a} \right)^{1/(\theta-1)}.$$

Each has productivity elasticity $-1/(\theta-1)$; the actual time endowment is fixed at one.

Proof. The share $s(k) = wk^\rho/(wk^\rho + 1 - w)$ strictly decreases from one to zero, and

$$\frac{ky'(k)}{y(k)} = \theta s(k), \quad \frac{ky''(k)}{y'(k)} = \rho - 1 + (\theta - \rho)s(k).$$

Thus $y(k)/k$ has its unique maximum H_a at k_a , and y' its unique maximum Φ_a at k_Φ . Both functions vanish at either end of the positive axis. This proves the three cases in (2.5), including the two roots above the threshold. Strict replacement at full labor remains strict at some labor below one by continuity. At a root, maintenance requires full labor and zero consumption. Outside the interval it is impossible. Homogeneity gives $F_p(K, L)/K \leq AH_a L^{\theta-1}$ and $F_{p,K}(K, L) \leq A\Phi_a L^{\theta-1}$. These inequalities give the contraction and the necessity of both capacity conditions. The two

capacity bounds are attained at $K = Lk_a$ and $K = Lk_\Phi$; when the respective condition is strict, some labor input below one satisfies it. Solving for the required labor inputs gives their formulas and elasticities. $\square$

A productivity loss raises both labor thresholds. These thresholds allow the capital–labor ratio to vary; they are not labor requirements at a fixed capital stock. Each strict test requires labor above its threshold. Their common elasticity $-1/(\theta - 1)$ makes this effect large when returns are close to constant. Available labor limits how far additional work can offset the loss.

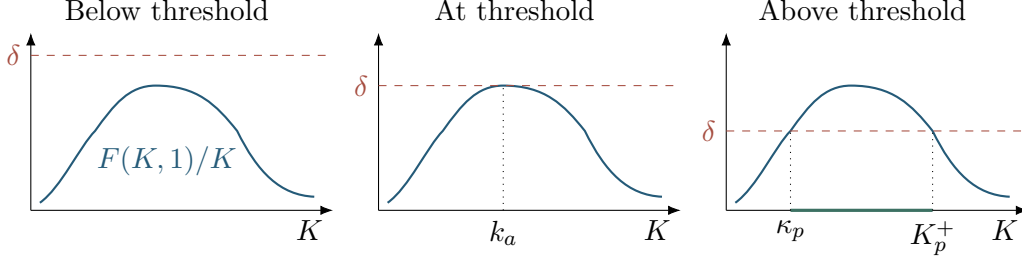


Figure 2: The physical sustainable set. The blue curve is output per unit of capital at full labor; the dashed line is depreciation per unit of capital. Maintenance is feasible where the curve reaches or exceeds the line. The set is empty, a singleton, or an interval. Its endpoints require full labor and zero consumption; its interior permits positive consumption and leisure. The curves are schematic, not computed examples. The upper root limits constant maintenance, not productive survival after drawing down a larger initial stock.

2.2 Physical and Euler return boundaries

The Euler return test has two boundary stocks, just as physical maintenance does. It asks whether an extra unit of investment can earn a marginal return above the rate required by discounting. Define its strict attainable set by

$$\mathcal{E}_p = \{K > 0 : \text{some } L < 1 \text{ satisfies } \beta[q + F_{p,K}(K, L)] > 1\}.$$

Since $F_{p,K}$ increases with labor, this set is empty when $A \leq A_{\text{ret}}$. When $A > A_{\text{ret}}$, it is the open interval (K_E^-, K_E^+) , whose endpoints solve $Ay'(K) = D$. At either endpoint the Euler return equals one only at full labor; neither endpoint passes the strict test. These are return boundaries, not steady states: the test does not impose the labor choice, consumption or global optimality conditions of a stationary allocation. This comparison applies throughout the primitive family (2.1); the optimal movements in Figure 1 also require the patience and productivity restrictions of Theorem 1.

Figure 3 shows all six configurations away from equalities. Favorable returns can exist before any stock is maintainable, or maintenance can exist before any favorable return is attainable. When both intervals exist, the return interval lies below, overlaps, or lies inside the maintenance interval. It always ends before the upper physical boundary. In particular, a favorable marginal return can coexist with output too low to replace depreciation. In panel (b), every feasible history tends to zero despite favorable Euler returns at the stocks in the blue interval: no stock produces enough to replace its depreciation. None of these tests locates the welfare frontier.

The equality cases complete the classification. At $A = A_{\text{rep}}$, the physical interval reduces to $\{k_a\}$; if the return interval exists, it lies strictly below k_a . At $A = A_{\text{ret}}$, the two Euler roots coincide at k_Φ , but the strict set remains empty. When both intervals have positive length, their lower physical boundary can meet either Euler boundary: $\kappa_p = K_E^+$ or $\kappa_p = K_E^-$. The Euler double root can also coincide with κ_p . Appendix A.3 gives the exact parameter conditions and proves that no other ordering is possible.

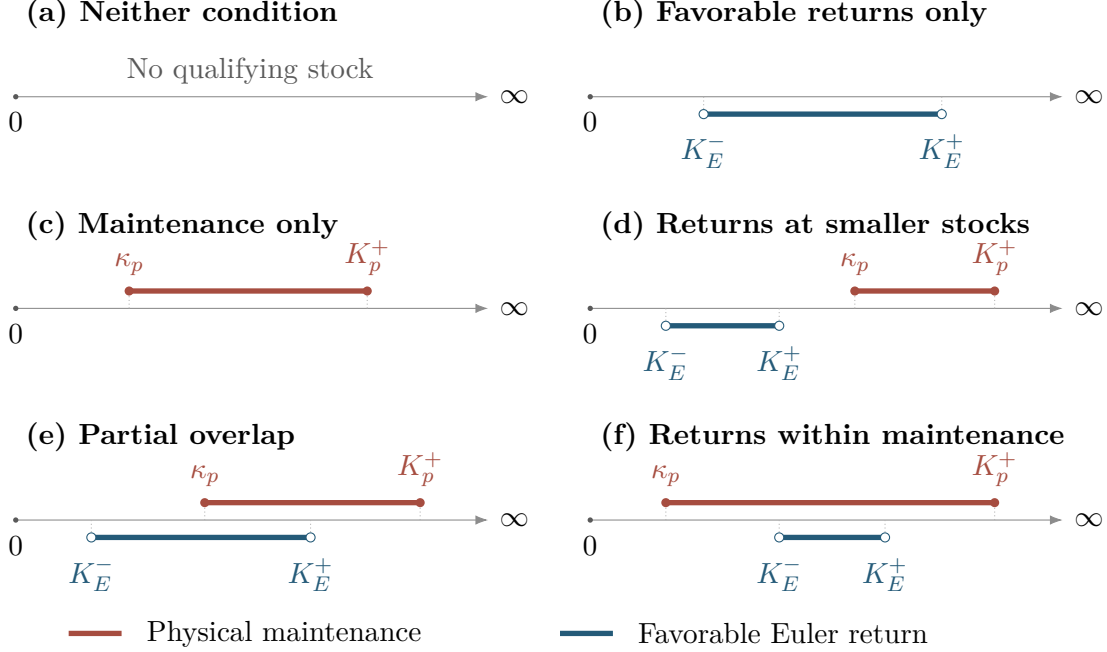


Figure 3: All strict orderings of physical maintenance and attainable Euler returns. Each half-line starts at zero and extends to arbitrarily large capital. Closed red endpoints permit maintenance only with full labor and zero consumption; blue endpoints are excluded from the strict return test. The ordering in each panel is analytical; distances are schematic. Axis arrows indicate the capital domain, not optimal accumulation. Optimal movements require the additional welfare classification shown in Figure 1. Equality cases are described in the text.

2.3 Productivity losses and the maintenance boundaries

A loss of usable output contracts the sustainable interval from both ends. At its lower boundary, a larger stock is needed to exploit complementarity. At its upper boundary, the available labor can support less capital. Near the productivity threshold, both boundaries are highly sensitive to the loss.

Proposition 2 (Physical vulnerability to productivity losses). *Fix all primitives except $A > A_{\text{rep}}$, and write $\kappa(A)$ and $K^+(A)$ for the physical boundaries. Each is differentiable and satisfies*

$$\frac{d \log K^*(A)}{d \log A} = -\frac{1}{\theta s(K^*(A)) - 1}, \quad s(K) = \frac{wK^\rho}{wK^\rho + 1 - w}. \quad (2.7)$$

The lower boundary falls with productivity, the upper boundary rises, and both elasticities diverge in absolute value as $A \downarrow A_{\text{rep}}$.

Let effective productivity fall to $A_e = (1 - \tau)A$, with $0 < \tau < 1$. If $A_e > A_{\text{rep}}$, then

$$\kappa(A) < \kappa(A_e) < k_a < K^+(A_e) < K^+(A).$$

Every stock in $(\kappa(A), \kappa(A_e)]$ previously permitted constant capital with positive consumption and leisure. After the loss, every history with finite discounted welfare from that band becomes extinct. If

$$\tau > \tau^* := 1 - \frac{A_{\text{rep}}}{A} = 1 - \lambda_{\text{rep}}^{\theta-1}, \quad (2.8)$$

every feasible history from every initial stock satisfies $K_t \leq (q + A_e H_a)^t K_0 \rightarrow 0$.

The proof is in Appendix A.4. At the new lower boundary, a feasible constant path still exists, but it requires zero consumption and full labor. The extinction conclusion includes this boundary because it concerns histories with finite discounted welfare. These comparative statics concern physical maintenance; they do not give the response of the welfare frontier to productivity.

2.4 Physical maintenance and optimal extinction

At $A = A_{\text{rep}}$, maintaining the singleton stock requires zero consumption and full labor. An optimum instead consumes a positive fraction of its capital. This consumption choice turns weak physical replacement into strict optimal decline.

Corollary 1 (Optimal extinction at the physical threshold). *If $A \leq A_{\text{rep}}$, there is an explicit constant $\gamma_p \in (0, 1)$ such that every optimum from every $K_0 > 0$ satisfies*

$$K_t \leq \gamma_p^t K_0 \rightarrow 0. \quad (2.9)$$

Appendix A.2 gives γ_p and proves the consumption bound independently of productive capture or classification. For $A < A_{\text{rep}}$, Proposition 1 already forces every feasible history to contract. At equality, the corollary concerns optimal histories: the zero consumption maintenance path remains feasible. Above the threshold, physical sustainability leaves the welfare decision open. Example 1 gives closed form physical boundaries in a harmonic specialization; the next section determines optimal preservation.

3 Minimum viable scale and welfare

3.1 A unique frontier in the CES economy

Under the patient condition (2.3), sufficiently high productivity makes productive accumulation worth its transition cost above a unique initial stock. The classification covers every optimal history. The stocks $b, h, \bar{K}(A)$ define regions used in the proof; optimal choices determine the welfare frontier.

Theorem 1 (Minimum viable scale). *Fix the six primitives other than A and suppose (2.3) holds. The explicit construction in (A.13)–(A.14) gives a finite A_* and stocks $0 < b < h < \bar{K}(A)$. For every $A > A_*$, there is a unique frontier*

$$\kappa_p < S_p < b. \quad (3.1)$$

Every optimum from $0 < K < S_p$ becomes extinct. Every optimum above S_p enters $[h, \bar{K}(A)]$ in finite time and remains there. At S_p , attained extinct and productive optima coexist and exhaust the optimum set; their first successors lie strictly below and above S_p , respectively.

For $\kappa_p < K < S_p$, every optimal successor is below K ; for $S_p < K \leq b$, every optimal successor is above K . Every extinct optimum from $0 < K \leq S_p$ has a strictly decreasing capital path.

The classification persists under sufficiently small changes in all seven primitives within the patient region, and the frontier varies continuously with those primitives.

On (κ_p, S_p) in Figure 1, constant capital with positive consumption and leisure is feasible, yet every optimum liquidates. Preservation does not repay the consumption and leisure it costs.

The proof connects the two regions. Physical constraints force extinction at low stocks; at high stocks, a feasible productive policy beats a regional bound on waiting and an independent bound on adverse continuations. Maximizing over current labor then orders optimal choices in the intervening region: a larger initial stock cannot induce a smaller optimal next period stock. The stationary equations rule out optimal reproduction of any stock between the two regions. Together these properties yield the unique frontier without assuming global concavity or global order of the optimal policy.

3.2 Choices and marginal welfare at the frontier

Equal welfare at the frontier conceals different current decisions. Next period's capital and labor jump upward. The local increasing differences property makes an extra unit of initial capital more valuable at a larger optimal successor. Productive continuations therefore have higher marginal welfare at the frontier. Figure 4 connects the separation of choices to the upward kink in welfare.

Theorem 2 (Frontier choices and welfare). *In an economy covered by Theorem 1, let $S = S_p$ and let $\mathcal{O}_-$ and $\mathcal{O}_+$ be its extinct and productive optimal history sets from S . These sets are compact. For each such history define $a_\pi = \eta\{q + F_{p,K}(S, L_0)\}/C_0$. Then*

$$\begin{aligned} \min_{\mathcal{O}_+} a_\pi &> \max_{\mathcal{O}_-} a_\pi, \\ V'_{p,-}(S) &= \min_{\mathcal{O}_-} a_\pi, \quad V'_{p,+}(S) = \max_{\mathcal{O}_+} a_\pi. \end{aligned} \tag{3.2}$$

In particular, writing

$$j_p := \min_{\mathcal{O}_+} a_\pi - \max_{\mathcal{O}_-} a_\pi, \quad J_p := V'_{p,+}(S) - V'_{p,-}(S),$$

one has $0 < j_p \leq J_p$. Every optimal successor selection jumps upward at S , with limits $\min G_p(S) < S < \max G_p(S)$. The corresponding labor limits are strictly ordered. No optimal history reaches S after its initial date.

At the frontier, every productive optimum assigns greater marginal value to capital than every extinct optimum. The left derivative selects the smallest extinct marginal value; the right derivative selects the largest productive one. Their difference is therefore positive even when either regime has several optimal histories.

Near S , successors from initial stocks below S lie at or below $\min G_p(S)$; those from stocks above S lie at or above $\max G_p(S)$. Closedness of the optimal successor correspondence fixes these one sided limits even when several histories are optimal at S .

The kink creates a local gain from reallocating capital. For two identical economies initially at S , the productive recipient values an additional unit more than the extinct donor values the unit it gives up. A small costless transfer $u > 0$ raises combined welfare by

$$V_p(S + u) + V_p(S - u) - 2V_p(S) = J_p u + o(u) > 0. \quad (3.3)$$

The recipient survives and the donor becomes extinct. The gain tolerates some loss in transit. If the donor gives up $(1 + \tau)u$ while the recipient receives u , the destroyed capital is τu , and

$$V_p(S + u) + V_p(S - (1 + \tau)u) - 2V_p(S) = [J_p - \tau V'_{p,-}(S)]u + o(u). \quad (3.4)$$

Since $0 < V'_{p,-}(S) < \infty$, every fixed $0 \leq \tau < J_p/V'_{p,-}(S)$ permits a gain for sufficiently small $u > 0$. This is a local aggregate welfare gain; a Pareto improvement would require compensation.

The strict separation of marginal values also controls the frontier's response to changes in fundamentals. Near the frontier, restricting the first successor to the lower or upper separated choice set gives two local welfare branches. Their values meet at the frontier with a strict slope gap. These are restrictions on the first decision, not welfare suprema over all eventually extinct or productive histories.

Proposition 3 (Local stability of minimum viable scale). *Fix a primitive vector covered by Theorem 1. There are a neighborhood $\mathcal{P}$ of that vector and a finite constant C_S such that*

$$|S_p - S_{\hat{p}}| \leq C_S \|p - \hat{p}\|, \quad p, \hat{p} \in \mathcal{P}. \quad (3.5)$$

The neighborhood allows changes in all seven primitives and lies inside the strict classification region.

This bound strengthens continuity without requiring a unique optimal history or a differentiable policy. It gives neither a signed comparative static nor a derivative of the frontier. The proof bounds primitive changes in normalized welfare and combines that bound with the persistent gap between the two branches' capital slopes.

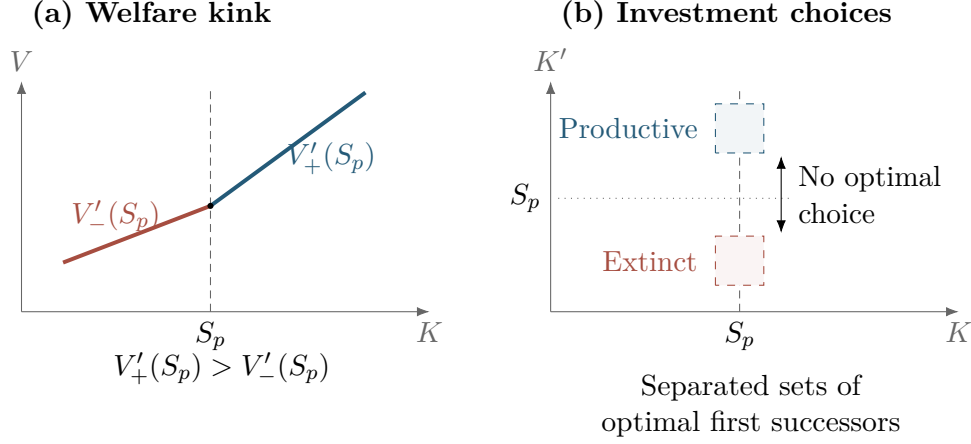


Figure 4: Welfare and investment at minimum viable scale. Panel (a) shows the one sided welfare slopes in Theorem 2. Investment in panel (b) is next period's capital. The panel uses containing bands for the two optimal successor sets, which need not be intervals. Both panels are schematic: axes indicate ordering, and the lines in panel (a) indicate tangent directions.

3.3 Economic activity in the two regimes

In a classified economy, an extinct optimum continues to work at sufficiently small positive stocks. Lemma 5 shows that every optimal action there uses positive labor, and every extinct optimum eventually enters that region. Production remains positive while capital tends to zero. Work improves current resources even when preservation does not repay its full transition cost.

A planner with positive capital can always stop working and consume the stock gradually. That option limits how much current consumption and leisure an optimum can sacrifice for the future. Proposition 7 in Appendix A.2 gives explicit positive floors for consumption relative to capital and for leisure at every optimal action. These bounds apply in both regimes, not only in the classified patient economies.

On an optimal productive tail, capital is bounded away from zero, so the relative consumption bound becomes an absolute positive floor. The resource constraint then bounds average output away from zero on every sufficiently long rolling window. The constants depend on the primitives and the productive interval; they are not uniform across economies. The output bound concerns rolling averages, not output at every date. Optimal productive survival thus sustains consumption and average production without requiring a stationary limit or a unique optimal policy.

4 First Passage Bellman Barriers

FPBB verifies optimal behavior without solving for all optimal choices. A feasible plan supplies a welfare floor, a regional barrier bounds perpetual waiting, and a separate ceiling bounds every continuation after an adverse action. Strict comparisons certify entry and permanence for every optimum even when other feasible histories avoid or leave the target. No equality between the bounds is required.

The selection theorem states this principle, and the CES application supplies its inputs directly from the primitives. Theorem 4 converts accumulated regional losses into a barrier with a positive welfare margin. The compact converse, Theorem 5, characterizes certificate existence and the

supremal margins. The renewal application shows why the block construction can succeed when the specified shorter comparisons cannot.

4.1 Why accurate welfare does not identify survival

At zero labor, capital can finance consumption without production. The policy $K_{t+1} = \beta q K_t$, $C_t = (1 - \beta)q K_t$ consumes the stock gradually and becomes extinct. We call it the cake policy. Its welfare, with primitive dependence indexed by p , is

$$E_p(K) = \frac{\eta}{1 - \beta} \log[(1 - \beta)qK] + \frac{\eta\beta}{(1 - \beta)^2} \log(\beta q). \quad (4.1)$$

The resource bound $R_p = q + AH_a$ limits the gain over cake uniformly across positive initial stocks.

Proposition 4 (Uniform approximation by extinction). *At every primitive vector in (2.1), optimal histories exist from every $K > 0$, and*

$$0 \leq V_p(K) - E_p(K) \leq \mathcal{D}_p = \frac{\eta}{(1 - \beta)^2} \log(R_p/q). \quad (4.2)$$

Following any optimum for n dates and then switching permanently to cake produces an extinct history $\pi^{[n]}$ satisfying

$$0 \leq V_p(K) - J_K(\pi^{[n]}) = \beta^n \{V_p(K_n) - E_p(K_n)\} \leq \beta^n \mathcal{D}_p. \quad (4.3)$$

Consequently the supremum of extinct welfare equals $V_p(K)$ at every positive stock.

For every $\epsilon > 0$, choosing a common switching date n and an optimum at each initial stock gives a family of extinct feasible plans with

$$\sup_{K>0} \{V_p(K) - J_K(\pi_K^{[n]})\} \leq \beta^n \mathcal{D}_p < \epsilon.$$

The bound follows from (4.3); no stationary or measurable policy selection is asserted. Even uniformly accurate initial welfare for a family of feasible plans does not certify survival. The claim concerns initial payoffs, not uniformly accurate Bellman equations or value approximations supplemented by action comparisons.

4.2 Regional verification of every optimum

Let $\mathbf{X}$ be a state set, $\mathbf{A}$ an action set, and $\mathcal{G} \subseteq \mathbf{X} \times \mathbf{A} \times \mathbf{X}$ the complete feasible transition graph. An edge $e = (x, a, y)$ earns $r(e) \in \mathbb{R} \cup \{-\infty\}$, discounted by $\beta \in (0, 1)$. Write $\Pi(x)$ for feasible infinite histories, $J_x(\pi)$ for their welfare, and $V(x) = \sup_{\pi \in \Pi(x)} J_x(\pi)$. Assume that discounted positive rewards are summable along each history. Welfare and its decomposition at any finite date are then well defined, including histories with welfare $-\infty$. No topology, concavity, ordering, or existence of an optimizer is imposed. The graph records every feasible action and its successor, not just the transitions of a chosen policy.

Fix disjoint staging and target sets M, H . A history is *core* if it stays in M until a finite first entry into H and then stays in H . Write $\mathcal{C}(x)$ for these histories from x ; entry occurs at date

zero when $x \in H$. Core behavior excludes departure and subsequent recovery, even if the history eventually remains in H . The adverse edges are

$$\begin{aligned} E_M &= \{(x, a, y) \in \mathcal{G} : x \in M, y \notin M \cup H\}, \\ E_H &= \{(x, a, y) \in \mathcal{G} : x \in H, y \notin H\}. \end{aligned} \tag{4.4}$$

The two classes cover departure before and after entry. Perpetual waiting in M takes neither kind of edge and needs its own bound.

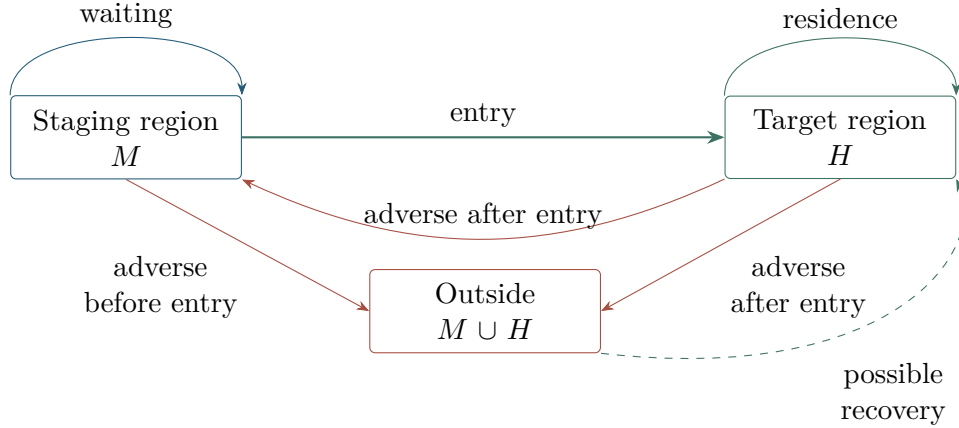


Figure 5: Entry and permanence require two comparisons. A feasible plan beats the regional bound on perpetual waiting, $W_M > B$. At the first adverse edge, a feasible plan earns more than the current reward plus its discounted continuation ceiling, $W_i > r + \beta \bar{V}$, where $i = M$ or H denotes the origin region. The latter comparison includes possible recovery. Arrows identify transition classes; a class need not be feasible in every problem.

Definition 1 (First passage certificate). A weak First Passage Bellman Barrier consists of a real function B on M , an independently established extended real ceiling $\bar{V} \geq V$ at every adverse successor, and finite welfare floors W_M, W_H satisfying the following conditions.

1. On every staying edge,

$$B(x) \geq r(e) + \beta B(y), \quad x, y \in M. \tag{4.5}$$

Every history with finite welfare staying in M satisfies $\liminf_{T \rightarrow \infty} \beta^T B(x_T) \geq 0$.

2. At every $x \in M$, a specified complete core history earns at least $W_M(x)$. At every $x \in H$, a specified history staying in H earns at least $W_H(x)$.
3. The comparisons hold on the complete feasible graph, including boundary actions:

$$\begin{aligned} W_M(x) &\geq B(x) & (x \in M), \\ W_M(x) &\geq r(e) + \beta \bar{V}(y) & (e \in E_M), \\ W_H(x) &\geq r(e) + \beta \bar{V}(y) & (e \in E_H). \end{aligned} \tag{4.6}$$

All extended real expressions must be defined. The certificate is *strict* if the three comparisons in (4.6) are pointwise strict. It has margin $\varepsilon \geq 0$ if every comparison slack is at least ε . A positive margin implies strictness.

A *regional Bellman barrier* is a real B satisfying the staying inequality and terminal condition. Its Bellman inequality may bind in a strict certificate. The witnesses establish feasible welfare, without requiring optimality. The ceiling covers all continuation after departure; a value restricted to histories that stay outside H is insufficient. It must also be established independently of the selection conclusion. When M is empty, the certificate verifies permanence in H alone.

Theorem 3 (Global selection). *A weak certificate makes every history from $M \cup H$ weakly dominated by a core history. Consequently, for $x \in M \cup H$,*

$$V(x) = \sup_{\pi \in \mathcal{C}(x)} J_x(\pi) = \sup_{\pi \text{ eventually remains in } H} J_x(\pi). \quad (4.7)$$

If an unrestricted optimum exists, a core optimum exists. A strict certificate makes every optimizer core. With common margin ε , staying in M forever loses at least ε ; an adverse event at date τ loses at least $\beta^\tau \varepsilon$ in initial welfare when its prefix has finite payoff.

Proof. A history with welfare $-\infty$ is dominated by the specified witness, which has finite welfare at the same initial state. Consider a history π with finite welfare that is not core. If it stays in M forever, iteration of (4.5) gives

$$\sum_{t < T} \beta^t r(e_t) \leq B(x_0) - \beta^T B(x_T).$$

The terminal condition implies $J_{x_0}(\pi) \leq B(x_0) \leq W_M(x_0)$. Otherwise there is a first adverse edge. Before that edge the history has stayed in M , or has entered H and remained there. Its payoff from the adverse date τ , including any subsequent recovery, is at most $r(e_\tau) + \beta \bar{V}(x_{\tau+1})$. Replace this edge and its tail by the specified witness at x_τ . The unchanged prefix and replacement form a core history with weakly greater welfare. Strict comparisons make each replacement strictly better. A core history already dominates itself. Core histories form a subset of the histories that eventually remain in the target, which in turn are unrestricted feasible histories. These inclusions give (4.7). Replacing an attained optimum by a weakly better core history preserves attainment. A common margin gives the stated loss for eternal waiting. An improvement at the adverse date raises initial welfare by that amount multiplied by β^τ . $\square$

4.3 Productive capture without solving for value

The productive anchor is established without computing the value function, the optimal policy or the welfare frontier. Here $x = K$, $a = (C, L)$, $y = K'$ and $r(e) = u_p(C, L)$: each edge records a feasible consumption, labor and investment decision. Lemma 4 constructs $0 < b < h < 1 < \bar{K}(A)$ and verifies the inequalities below from the primitives.

1. Regions and transition coverage. Let $\Phi(K) = qK + F_p(K, 1)$ and define the resource ceiling

$$R_{\text{cap}} = q\bar{K}(A) + A(1 - w)^{\theta/\rho}.$$

The construction ensures $\Phi(\bar{K}(A)) < R_{\text{cap}} < \bar{K}(A)$, so $[0, \bar{K}(A)]$ is invariant. Take $M = [b, h]$ and $H = [h, \bar{K}(A)]$. The certificate covers every feasible labor choice in $[0, 1]$. Waiting edges have both stocks in M ; adverse edges leave $M \cup H$ from M , or leave H from H . Invariance rules out successors above $\bar{K}(A)$.

2. A feasible entry and permanence plan. Set labor at $\sigma = 3/4$ and put $Z(K) = qK + F_p(K, \sigma)$. The plan chooses next period's capital to be one and then maintains it there, earning

$$W(K) = \eta \log[Z(K) - 1] + (1 - \eta) \log(1 - \sigma) + \frac{\beta}{1 - \beta} \{ \eta \log[F_p(1, \sigma) - \delta] + (1 - \eta) \log(1 - \sigma) \}. \quad (4.8)$$

The productivity conditions make its consumption positive at b . Resources increase with the initial stock, so the same investment and continuation remain feasible at every larger stock. Since $1 \in H$, this plan supplies both core witness floors, $W_M = W|_M$ and $W_H = W|_H$.

3. Unrestricted continuation after departure. On $[0, \bar{K}(A)]$, use the independent ceiling

$$\bar{V}(y) = \eta \log \Phi(y) + \frac{\beta \eta}{1 - \beta} \log R_{\text{cap}}, \quad \bar{V}(0) = -\infty. \quad (4.9)$$

Current consumption from y is at most $\Phi(y)$, every later consumption is at most R_{cap} , and leisure utility is nonpositive. These facts bound every continuation, including recovery. The ceiling discards leisure costs and almost all restrictions on future choices. Such a loose bound can suffice because the certificate requires a strict comparison at adverse actions, not an accurate value at every state.

4. Perpetual waiting and its terminal term. On M , set

$$B_M(K) = \eta \log \Phi(K) + \frac{\beta \eta}{1 - \beta} \log \Phi(h). \quad (4.10)$$

On every staying edge, $C \leq \Phi(K)$ and $K' < h$, while leisure utility is nonpositive. Hence

$$B_M(K) - u_p(C, L) - \beta B_M(K') \geq \beta \eta \log \frac{\Phi(h)}{\Phi(K')} \geq 0.$$

The barrier is bounded on M , so its discounted terminal term vanishes.

5. Strict comparisons and productive capture. Put $\lambda = \beta^2 \eta \log[R_{\text{cap}}/\Phi(h)] > 0$. Lemma 4 verifies the following margins, covering every adverse edge:

$$\begin{aligned} W(K) - B_M(K) &\geq \varepsilon_0 + \frac{\lambda}{1 - \beta} & (K \in M), \\ W(K) - \{u_p(C, L) + \beta \bar{V}(K')\} &\geq \varepsilon_0 & (e \in E_M \cup E_H), \\ \varepsilon_0 &= \frac{\eta}{2(1 - \beta)} > 0. \end{aligned} \quad (4.11)$$

These bounds give a full certificate with margin ε_0 . The first comparison excludes perpetual waiting; the second excludes every adverse departure, even when recovery would follow. Theorem 3 therefore establishes entry and permanence on the cover.

The witness comparison in Lemma 4 also rules out every optimal successor at or below h from any $K \geq b$, not merely adverse edges on the cover. Entry from M is immediate in this application. For initial stocks above $\bar{K}(A)$, global exclusion of low successors prevents a fall below the target, while resource contraction gives finite entry into the invariant cover. Both arguments are needed.

The certificate establishes productive capture from resource bounds and one fixed labor and investment plan. Local order and exclusion of stationary optima connect that anchor to extinction and locate the frontier below b . The next theorem recovers this barrier with $N = 1$ and handles losses that become uniformly positive only over longer blocks.

4.4 Calibrated construction from finite block losses

A loss measured against a regional reference may vanish at an individual decision even though it accumulates over every sufficiently long staying path. A test demanding strict loss at every decision would then fail. The construction uses this accumulated loss to lower the reference to a regional barrier. Calibration compares that reduction with the shortfall of a feasible core plan's welfare from the reference. The difference is the guaranteed staying margin. The construction gives the best such guarantee permitted by the block and calibration data.

Let $M \neq \emptyset$. Take a bounded regional reference $U : M \rightarrow \mathbb{R}$ whose residuals are nonnegative on every staying edge:

$$d_U(e) = U(x) - r(e) - \beta U(y) \geq 0, \quad x, y \in M. \quad (4.12)$$

Allow $d_U(e) = +\infty$ when $r(e) = -\infty$. The welfare objects have different roles. A valid certificate satisfies

$$B \leq W_M \leq V \quad \text{on } M, \quad W_H \leq V \quad \text{on } H,$$

while $V \leq \bar{V}$ at adverse successors. The barrier bounds perpetual waiting, the witnesses establish attainable welfare, and the ceiling bounds unrestricted continuation after departure. The reference U supplies the construction $B = U - \ell_U$ below; it need not bound unrestricted value or extend beyond M .

Fix an integer $N \geq 1$ and a finite $\Delta_{\text{blk}} > 0$. Suppose every N edge staying prefix satisfies

$$\sum_{t < N} \beta^t d_U(e_t) \geq \Delta_{\text{blk}}. \quad (4.13)$$

The test includes prefixes that cannot be extended indefinitely; an empty prefix class satisfies it.

To construct the correction, consider an auxiliary calculation of accumulated reference losses. At each step it can stop at a fixed cost or continue along a staying edge. This stopping option keeps the calculation bounded, including where no continuation is available. It is a construction device, not target entry or an additional economic action; the feasible graph is unchanged. For a bounded real function z on M and $x \in M$, define

$$C_N = \frac{\Delta_{\text{blk}}}{1 - \beta^N}, \quad c_N = C_N \beta^{1-N}, \quad (4.14)$$

$$(\mathcal{F}_U z)(x) = \min \left\{ c_N, \inf_{\substack{e=(x,a,y) \in \mathcal{G} \\ y \in M}} [d_U(e) + \beta z(y)] \right\},$$

where an empty infimum is $+\infty$. The clipping cost c_N is paid if the auxiliary calculation stops early. Iteration starts from the terminal cost C_N , the guaranteed correction to the reference. Over N steps, the chosen cap makes every early stop cost at least C_N after discounting; the block bound gives the same lower bound for a full prefix including its terminal cost. At a dead end the

cap remains available, and infinite residuals cannot lower an infimum. Function inequalities and maxima are pointwise.

Theorem 4 (Calibrated finite block construction). *Set*

$$g_0 \equiv C_N, \quad g_j = \mathcal{F}_U^j g_0, \quad \ell_U = \max_{0 \leq j < N} g_j, \quad B = U - \ell_U. \quad (4.15)$$

Then $C_N \leq \ell_U \leq c_N$, and B is a bounded regional Bellman barrier. If floors supported by specified feasible core histories satisfy $W_M \geq U - \alpha$, with $0 \leq \alpha < C_N$, its stay margin is at least $C_N - \alpha$. Specified target witnesses and independently established adverse comparisons of margin $\varepsilon_{\text{ad}} > 0$ complete a uniform certificate of margin $\min\{C_N - \alpha, \varepsilon_{\text{ad}}\}$. For every $0 \leq \alpha < C_N$, both margin bounds are sharp given only $N, \beta, \Delta_{\text{blk}}, \alpha, \varepsilon_{\text{ad}}$.

The construction and sharpness proofs are in Appendix C.1.

The block bound makes the N th iterate no smaller than the starting constant. Monotonicity then makes the operator applied to the finite maximum no smaller than that maximum. This is the inequality needed for the correction: subtracting ℓ_U from U gives the staying Bellman inequality, and boundedness makes the terminal term vanish. Calibration subtracts the witness shortfall α from the guaranteed correction C_N , leaving the staying margin $C_N - \alpha$. No larger uniform margin follows from the same block, calibration and adverse margin data alone, although additional information about a particular problem may yield a larger margin.

Calibration is essential: α measures the shortfall of feasible core welfare from the reference. A zero reward staging self loop and free entry into a zero reward target have value zero. The reference $U = 1$ has residual $1 - \beta$ and gives $C_N = 1$, but the smallest admissible calibration error is $\alpha = 1$. Waiting forever is still optimal. Replacing U by $U + u$, for $u \geq 0$, increases every block reference loss by $(1 - \beta^N)u$. The same certified bounds become $C_N + u$ and $\alpha + u$, leaving $C_N - \alpha$ unchanged. The construction thus measures an attainable welfare gain, not a gain created by choosing a higher reference level.

Recovering the CES barrier. For the barrier in (4.10), choose

$$U(K) = B_M(K) + \frac{\lambda}{1 - \beta} + \varepsilon_0, \quad N = 1, \quad \Delta_{\text{blk}} = \lambda + (1 - \beta)\varepsilon_0.$$

The direct staying inequality gives $d_U \geq \Delta_{\text{blk}}$, and (4.11) gives $W \geq U$, so calibration error is zero. The theorem has $C_1 = c_1 = \Delta_{\text{blk}}/(1 - \beta)$ and $\ell_U = C_1$. It therefore returns precisely $U - C_1 = B_M$ from the independent inputs of the CES certificate. Longer blocks are unnecessary for this productive anchor.

Summing block losses already bounds eternal staying welfare by $U - C_N$; the construction additionally enforces every staying Bellman inequality, including where immediate loss vanishes. No minimizing action is required. Constructing ℓ_U takes $N - 1$ applications of $\mathcal{F}_U$ over the complete controlled staying graph; the continuous action infima must still be bounded or solved analytically.

4.5 Limits of constant barriers and shorter constant reference tests

A machine can operate cheaply for several periods before maintenance becomes costly. Renewal avoids future upkeep but requires a current outlay. A bound based only on ordinary operating costs can therefore miss the reason to renew. The following separate economy tests the general method; it does not change the CES model.

There are $N \geq 2$ ordered wear states $M = \{0, \dots, N-1\}$ and a renewed target $H = \{h\}$. Rewards are net of a common operating flow. At wear state i , ordinary operation costs $s > 0$ and moves to any allowed $j > i$. Maintenance costs $c_e \geq c > s$ and moves to an allowed $j \in M$. Each state has a finite menu with at least one staying action; at least one ordinary operation is available.

Renewal costs $I_i > 0$ and moves to h , where continued operation has zero net reward. Quitting from M costs a and moves to outside activity z ; departure from h to z costs $d > 0$. At z , remaining earns zero and reopening h costs $f > 0$. Recovery is feasible, not excluded by definition.

Write $I_{\min} = \min_i I_i$, $\bar{I} = \max_i I_i$, and define

$$T_s = \frac{s}{1-\beta}, \quad D_N = \frac{c-s}{1-\beta^N}, \quad C_N = T_s + \beta^{N-1}D_N. \quad (4.16)$$

Assume

$$T_s < I_{\min} \leq \bar{I} < C_N, \quad a > \bar{I}. \quad (4.17)$$

These restrictions define a nonempty open parameter region. Renewal costs more than the present value of ordinary operation alone, but less than the cost that recurring maintenance makes unavoidable. A *constant reference test* applies Theorem 4 with $U \equiv u$, a uniform block loss bound and a uniform feasible welfare calibration. Success requires a positive calibrated margin.

Proposition 5 (Renewal and the limits of constant barriers). *Under (4.17), the independent inputs*

$$\begin{aligned} B(i) &= -T_s - D_N\beta^{N-1-i}, & W_M(i) &= -I_i, & W_H(h) &= 0, & \bar{V} &= 0, \\ \varepsilon &= \min\{C_N - \bar{I}, a - \bar{I}, d\} > 0 \end{aligned} \quad (4.18)$$

form an FPBB of margin at least ε . Optima exist, and every optimum from $M \cup H$ enters h and remains there. From a wear state, entry takes at most N transitions; this bound is sharp. No constant regional barrier can satisfy even weak witness domination. No calibrated single period test with a constant reference succeeds.

For the periodic subcase, let $I_i = I$, let ordinary operation move i to $i+1$, and let the last wear state offer maintenance back to zero at cost c . These are the only staying actions. Every optimum waits exactly $N-1-i$ periods and then renews. Its value is

$$V(i) = -T_s - \beta^{N-1-i}(I - T_s), \quad V(h) = V(z) = 0. \quad (4.19)$$

Every calibrated constant reference test of horizon $n < N$ fails, whereas the N period test succeeds.

Every N staying decisions include maintenance: N strict increases among N wear states are impossible. With $U = 0$, every staying block has loss at least

$$\Delta_{\text{blk}} = (1 - \beta^N)T_s + \beta^{N-1}(c - s) = (1 - \beta^N)C_N, \quad \alpha = \bar{I} < C_N.$$

The bound on α gives the feasible welfare calibration. Theorem 4 therefore supplies a barrier with the stated staying margin, as does the explicit formula above. Finite state structure sharpens the timing conclusion:

$$\tau_H := \min\{t \geq 0 : x_t \in H\} \leq N. \quad (4.20)$$

An optimum cannot revisit a wear state. Its intervening cycle would have optimal continuations at both ends, so repeating that cycle forever would also be optimal, contradicting selection. This uses the familiar path and cycle structure of deterministic decision problems (Post and Ye,

2015, Section 3), but applies to every optimal history. The feasible graph still allows perpetual maintenance. The periodic subcase attains the bound from state zero. Appendix C.2 gives the full proof.

The certificate uses only the wear order, cost bounds and immediate renewal plans; it need not identify which maintenance or renewal action is optimal. In the periodic subcase, immediate renewal is strictly suboptimal before the last wear state, yet it remains a valid witness. The adverse ceiling need not equal value: replacing zero by any constant ζ with $0 < \beta\zeta < \min\{a - \bar{I}, d\}$ reduces each adverse margin by only $\beta\zeta$. All comparisons remain strict.

The failure of constant barriers is structural. An ordinary operating edge requires a constant barrier to be at least $-T_s$. Every feasible core history from a wear state earns strictly less than $-T_s$, because renewal costs more than ordinary operation forever. No choice of constant barrier or feasible witness can close this gap. In the periodic subcase, any shorter block admits a prefix containing only ordinary operation; raising the constant reference cannot help because it raises the calibration shortfall by the same amount. The full block captures unavoidable upkeep.

The example also shows why positive exact Bellman loss at every staying decision is too demanding: those losses vanish on optimal waiting edges. The constructed state dependent barrier nevertheless satisfies a single period Bellman inequality. The failed constant reference tests do not rule out other state dependent methods. The exact solution exposes the separation; it is not used to construct the certificate.

Immediate access to renewal is not essential. Suppose renewal is available only at a nonempty set $R \subseteq M$, at costs $I_r > T_s$. From each state i , specify a feasible path of k_i ordinary operations to an eligible state $r_i \in R$. Strictly increasing wear gives $0 \leq k_i \leq N - 1 - i$. The plan then renews and remains at h . Its discounted cost is

$$Q_i = T_s + \beta^{k_i}(I_{r_i} - T_s), \quad \gamma = \min_i \{D_N \beta^{N-1-i} - \beta^{k_i}(I_{r_i} - T_s)\}. \quad (4.21)$$

These paths are specified witnesses, not candidate optimal policies.

Corollary 2 (Delayed access and state dependent calibration). *Retain the ordinary, maintenance and outside options of the renewal economy, with renewal eligibility and access paths as above. If $\gamma > 0$ and $a > \max_i Q_i$, the barrier in (4.18), witness floors $W_M(i) = -Q_i$ and $W_H(h) = 0$, and ceiling $\bar{V} = 0$ form an FPBB with margin at least*

$$\min\{\gamma, a - \max_i Q_i, d\} > 0.$$

Optima exist. Every optimum enters h within N transitions and remains there. No constant regional barrier can satisfy weak witness domination.

For example, let renewal be available only at the last wear state, at cost I , and include each consecutive ordinary transition. The specified path has $k_i = N - 1 - i$, so

$$\gamma = \beta^{N-1}(T_s + D_N - I) > 0 \quad \text{if} \quad T_s < I < T_s + D_N.$$

Together with $a > I$, this certifies the additional interval $C_N \leq I < T_s + D_N$. The gain comes from calibrating each delayed witness to its own state, not from restricting renewal access. The same comparison works when additional renewal actions are available. In the pure periodic cycle with renewal only at the last state and $a > I$, $T_s + D_N$ is the exact upper threshold for selection of every optimum: at equality, perpetual maintenance ties renewal. Thus access can be delayed,

and the outlay can exceed the original uniform cost bound, without solving for value. This wider region does not imply that the constant reference N period test succeeds there.

4.6 What positive margins protect

A large loss far in the future can have a small effect on initial welfare. A continuation comparison measures the loss at the date of the decision. Positive certificate margins give an accuracy guarantee for these comparisons and a bound on permissible payoff changes. They therefore extend the method from exact optimizers to histories whose continuations stay sufficiently close to the feasible witness floors.

Proposition 6 (Continuation accuracy and sharp reward robustness). *Let a uniform FPBB have margin $\varepsilon > 0$.*

1. *A history with finite welfare from $M \cup H$ is core if, at every visited state in M or H , its actual continuation satisfies*

$$J_{x_t}(\pi_{t:}) \geq W_i(x_t) - \kappa, \quad 0 \leq \kappa < \varepsilon. \quad (4.22)$$

Here W_i is the corresponding floor W_M or W_H . If V is finite at every visited state in $M \cup H$, continuation regret $V(x_t) - J_{x_t}(\pi_{t:}) \leq \kappa < \varepsilon$ at every such date suffices. The witness test in (4.22) needs no value function.

2. *Keep the graph and β fixed, preserve all $-\infty$ rewards, and change finite rewards to $\tilde{r} = r + \psi$, with $\underline{\psi} \leq \psi \leq \bar{\psi}$. Put $\omega = \bar{\psi} - \underline{\psi}$. If $\omega \leq (1 - \beta)\varepsilon$, the shifted inputs*

$$\tilde{B} = B + \frac{\bar{\psi}}{1 - \beta}, \quad \tilde{V} = V + \frac{\bar{\psi}}{1 - \beta}, \quad \tilde{W}_i = W_i + \frac{\underline{\psi}}{1 - \beta} \quad (4.23)$$

give a certificate with margin at least $\varepsilon - \omega/(1 - \beta)$. Every existing perturbed optimizer is core when the inequality is strict. Both strict thresholds are sharp over the stated certificate class.

The proof is in Appendix C.3. Adding the same constant to every reward leaves the margin unchanged. Under $|\psi| \leq \zeta$, the sufficient condition is $\zeta < (1 - \beta)\varepsilon/2$. The payoff estimate is standard; the certificate converts it into a bound on changes that preserve entry and permanence.

For the CES certificate, $\varepsilon_0 = \eta/[2(1 - \beta)]$. Write $\text{osc } \psi$ for the supremum of ψ minus its infimum over all finite reward edges of the complete feasible graph. Consequently any bounded additive reward change with

$$\text{osc } \psi < \eta/2 \quad (4.24)$$

preserves productive entry and permanence for every existing optimizer from $K \geq b$. The same shift preserves the uniform comparison excluding successors at or below h , including from stocks above $\bar{K}(A)$; unchanged resource contraction then gives entry into the cover. This conclusion concerns capture, not preservation of the unique frontier, local order or welfare kink under arbitrary payoff changes. It does not assert existence after a discontinuous perturbation.

4.7 Completeness and the best certificate margins

The preceding results construct sufficient certificates. Under the compactness and attainment assumptions below, the certificate class is also complete: optimal selection implies a strict certificate.

Its best uniform margin has an exact interpretation. Perpetual waiting and adverse departure are the only two ways to violate the target behavior, and their smallest losses determine the supremal margin.

Give states and actions metric topologies. Suppose $M \neq \emptyset$, V is real and attained on $M \cup H$, and $V|_M$ is bounded. At adverse successors, assume V is real or $-\infty$, with every finite value attained. For each feasible edge $e = (x, a, y)$ with $x \in M \cup H$, define exact Bellman loss by

$$d(e) = V(x) - \{r(e) + \beta V(y)\} \in [0, +\infty].$$

Assume the complete staying graph $\mathcal{K} = \{(x, a, y) \in \mathcal{G} : x, y \in M\}$ is compact and d is lower semicontinuous there. For a finite staying path P , put

$$\begin{aligned} \mathcal{L}_n(P) &= \sum_{t < n} \beta^t d(e_t), & \delta_n &= \inf_{|P|=n} \mathcal{L}_n(P), \\ G &= \inf_{\pi \text{ stays in } M} \sum_{t \geq 0} \beta^t d(e_t), & a_* &= \inf_{e \in E_M \cup E_H} d(e). \end{aligned} \tag{4.25}$$

Here d measures loss on one edge and $\mathcal{L}_n$ on one prefix. The infima δ_n , G and a_* range over all prefixes of length n , perpetual staying histories and adverse edges, respectively. Unlike the reference residual d_U , these losses use the exact value function. All initial states in M enter the staying infima. Empty infima are $+\infty$, and $\delta_0 = 0$.

The assumptions have separate roles. Compactness of the staying graph and lower semicontinuity of loss produce limiting paths without increasing their loss. Bounded regional value makes discounted terminal values vanish. Attainment at adverse successors ensures that a zero loss adverse edge can be followed by an optimal continuation, which would contradict selection.

Theorem 5 (Compact completeness and sharp margins). *Under these assumptions:*

1. *Every optimizer from $M \cup H$ is core if and only if a pointwise strict FPBB exists. Selection implies a bounded regional barrier with a common positive staying margin. If the adverse graph is also compact and its loss is lower semicontinuous, selection is equivalent to a uniform strict FPBB.*
2. *Suppose selection holds. Let Γ_M be the supremum of uniform staying margins with witness floor $V|_M$, and let Γ be the supremum of full certificate margins over all valid barriers, witness floors, and continuation ceilings, including weak certificates. Then*

$$\Gamma_M = G, \quad \Gamma = \min\{G, a_*\}. \tag{4.26}$$

Every finite staying margin $0 \leq \gamma < G$ has a bounded barrier.

Appendix C.4 applies the calibrated block construction to the compact loss profile. The converse turns absence of an optimal eternal stayer into a positive finite block loss and then into a bounded barrier. It characterizes the unrestricted certificate class, not the success of a chosen barrier template. The margin formula also identifies the binding obstruction: improving the staying barrier cannot raise the full margin above a_* , and improving adverse bounds cannot raise it above G . The formula gives a supremum, not endpoint attainment. If $a_* = 0$, every optimum can be core while the full uniform margin is zero.

Exact value serves here to characterize the certificate class. The economic applications construct certificates from independent bounds and feasible witnesses. Labor lying in $[0, 1]$ alone does not verify the compact converse assumptions for the logarithmic CES economy.

4.8 The methodological contribution and its scope

Selection, calibrated construction and compact completeness have different antecedents. The comparisons below identify what each result adds to the established verification arguments.

Behavior from independent bounds. Bellman inequalities and first exit replacement are established (Akao, Ishii, Kamihigashi, and Nishimura, 2022, Proposition 2.3 in the discussion paper). FPBB formulates their joint obligations on a complete controlled graph, including indefinite delay, departure before entry, departure after entry, and recovery. The selection theorem needs no topology, global concavity or global ordering. Singular rewards are allowed, with a separate terminal condition governing the regional barrier.

The closest global dual comparison is Gaitsgory, Parkinson, and Shvartsman (2019, Theorem 2.1 and Proposition 3). Their certificate can differ from value: any global dual maximizer suffices. Normalization yields a global Bellman bound exact at the initial state. FPBB does not require such a maximizer or exact contact. The regional barrier, feasible witness floors and unrestricted continuation ceiling can remain distinct. The fixed CES labor plan and the suboptimal renewal witness demonstrate this separation directly.

Construction with a sharp welfare guarantee. Finite step Lyapunov theory converts block decay into a single period inequality. Theorems 19 and 20 of the author version of Geiselhart, Gielen, Lazar, and Wirth (2014) give finite sum and maximum constructions for a deterministic map with norm comparison bounds. Theorem 4 uses the same finite iterate principle on a discounted controlled graph, but its conclusion is a welfare certificate. Clipping keeps the construction bounded at dead ends and infinite losses; calibration relates it to attained feasible welfare. The guaranteed full margin $\min\{C_N - \alpha, \varepsilon_{\text{ad}}\}$ is sharp for every admissible calibration error. The renewal example proves failure of entire constant and shorter constant reference classes, not just of one candidate within those classes.

A converse for optimal entry and permanence. Discounted dissipativity already has normalization, storage constructions and converse results. With bounded state set and stage cost, Grüne, Müller, Kellett, and Weller (2021, Theorem 5.4) characterizes robust optimality of a reference equilibrium by discounted strict dissipativity with bounded storage; Theorem 6.2 characterizes storage through normalized required supply under reachability from that equilibrium. Theorem 5 instead concerns all optimal histories entering and remaining in a prescribed region. It needs no reference equilibrium or claim of convergence to one. Under its own compactness and attainment assumptions, it identifies both certificate existence and the exact margin identities $\Gamma_M = G$ and $\Gamma = \min\{G, a_*\}$. The proof links optimal selection to a positive block loss and the calibrated construction; those conclusions do not follow merely from a discounted storage inequality.

Selection rather than feasible reachability. Meng and Liu (2022, Definition 2.9 and Theorems 4.24 and 4.27) give robust reach and stay and Lyapunov–barrier characterizations for the solutions of specified hybrid dynamics. FPBB selects every welfare optimizer from a feasible graph that also admits histories violating the target behavior. The renewal graph admits perpetual cycling, and cake can leave the CES productive target. Thus the conclusion here is neither that all feasible histories succeed nor simply that one successful control exists. The witness establishes feasibility; welfare comparisons establish selection.

Other state dependent Bellman constructions, finite step Lyapunov methods and exact dynamic programming remain available. FPBB can establish the specified behavior without recovering complete value or an optimal policy, but finding suitable independent bounds may still be difficult. The advantage established here concerns the information sufficient for verification, not general algorithmic complexity.

5 Where the specified criteria fail

Physical maintenance asks whether production can replace depreciation. An attainable return screen asks whether some labor choice makes the discounted marginal capital return exceed one. Neither accounts for the consumption and leisure costs of the optimal transition. The next economy satisfies both screens on an interval where every optimum liquidates. In the same economy, global concavity and increasing differences fail for the return after optimizing over the full labor set. Positive consumption and leisure at an optimum do not make this return concave. Write $s = K'$ when treating successor capital as an argument of that return. For $K > 0$ and $0 \leq s < \Phi(K) = qK + F_p(K, 1)$, define

$$r_p(K, s) = \max_{\substack{0 \leq L \leq 1 \\ qK + F_p(K, L) > s}} \{ \eta \log[qK + F_p(K, L) - s] + (1 - \eta) \log(1 - L) \}. \quad (5.1)$$

Fix $q = \eta = w = 1/2$, $\rho = -1$, $\theta = 3/2$, and $4/5 < \beta < 1$. Let $A_*^{\text{gen}}(\beta)$, b_β , h_β be the classification threshold and targets with resource cover $8A$. Put

$$\begin{aligned} d_\beta &= 2/\beta - 1, & k_D &= \frac{3/2 - d_\beta}{d_\beta}, & c_D &= \frac{8d_\beta \sqrt{3/2 - d_\beta}}{3\sqrt{3}}, \\ A_*^{\text{sep}}(\beta) &= \max\{10, A_*^{\text{gen}}(\beta), 1/(2c_D)\}. \end{aligned} \quad (5.2)$$

Theorem 6 (Favorable screens and global return failures). *For $A > A_*^{\text{sep}}(\beta)$, every optimum is classified by Theorem 1, with productive target $[h_\beta, 8A]$. There exist a fixed labor choice $0 < L_\dagger < 1$ and a nonempty open interval $I_p \subset (\kappa_p, S_p)$ such that at every $K \in I_p$,*

$$F_A(K, L_\dagger) > K/2, \quad \beta[1/2 + F_{A,K}(K, L_\dagger)] > 1, \quad (5.3)$$

yet every optimum crosses below κ_p in finite time and becomes extinct. Constant capital with positive consumption and leisure is feasible at these stocks.

In the same economy, the actual labor maximized return has an upward kink in saving and violates concavity in saving and increasing differences. Explicit comparison stocks lie inside $(h_\beta, 8A)$. Classification, the extinct interval with both favorable screens, and the two strict finite return violations persist on an open neighborhood in the seven primitives.

The return in (5.3) is attained at an interior feasible labor choice; it is not an Euler condition along an optimum. The proof constructs that labor choice and an extinct interval analytically. The global return comparisons likewise maximize over all labor choices. The saving kink arises in that envelope, rather than in a selected stationary branch. Appendix D.1 provides both constructions.

Increasing differences in Assumption 1(iii) of the June 5, 2026 version of Huang, Pouzo, and Rodríguez-Clare (2026) is therefore unavailable on the productive cover. Regional verification and

Table 1: What the analytical comparisons establish

Comparison	Demonstrated limitation	Established conclusion
Maintenance and attainable returns	Both hold on an interval where every optimum becomes extinct.	In the harmonic CES economy, capture plus local order distinguishes physical capacity from optimal preservation.
Global concavity and increasing differences	Both fail for the actual labor maximized return. Additive state and successor potentials cannot repair the crossed difference.	The same economy is classified by regional verification and local order.
Initial welfare accuracy	Extinct plans approximate optimal welfare uniformly over positive stocks.	FPBB excludes indefinite waiting and adverse departure through conditional comparisons.
Constant barriers and shorter constant reference tests	No constant barrier works in the renewal economy; every calibrated constant reference test with $n < N$ fails in its periodic subcase.	An explicit state dependent barrier and the N period construction certify renewal and permanence.
Complete value or policy recovery	Not shown to fail; it answers more than the behavioral question requires.	Both applications use independent bounds. Witnesses can be suboptimal and need not meet continuation ceilings.

The first two rows concern different stocks in the same harmonic economy (Theorem 6). The third is Proposition 4; the fourth is Proposition 5. The final row concerns both applications and is an information comparison, not an impossibility result.

local order nevertheless yield classification. Failure of this global hypothesis does not imply that the optimal transition is globally nonmonotone.

Nor can an additive normalization repair this order obstruction. For any $\tilde{r}(K, s) = r_p(K, s) + a(K) + b(s)$, the crossed difference on a feasible rectangle is unchanged:

$$\Delta\tilde{r} = \Delta r_p, \quad \Delta r = r(K_2, s_2) - r(K_2, s_1) - r(K_1, s_2) + r(K_1, s_1).$$

The terms a and b cancel. In particular, cake normalization $a = -E_p$, $b = \beta E_p$ preserves the strictly negative difference. This is an algebraic obstruction to global increasing differences, not a claim that every transformation or verification method fails.

6 Conclusion

Physical maintenance does not settle whether a productive base is worth preserving. Joint increasing returns can reward expansion of capital and labor, while saving and work sacrifice consumption and leisure. In the patient CES family at sufficiently high productivity, a unique welfare frontier separates optimal extinction from productive survival. It lies strictly above the lower physical maintenance boundary. From an interval of stocks that permit maintenance with positive consumption and leisure, every optimum instead liquidates capital. At the frontier, both regimes attain the same welfare, but productive continuations choose more capital for the next

period and more labor, and have higher marginal welfare. This separation creates a local gain from reallocating capital between identical economies at the frontier, even with some loss in transit. It also makes the frontier locally Lipschitz in the primitives.

Extinct histories can nevertheless approximate optimal initial welfare. First Passage Bellman Barriers resolve the behavioral question through two comparisons: feasible plans dominate perpetual waiting and adverse departure, even when recovery remains possible. The CES certificate establishes productive capture without computing the full value function; local order and exclusion of stationary optima then locate the unique frontier.

The renewal economy shows why construction matters: constant barriers fail, and its periodic subcase defeats every calibrated constant reference test shorter than the wear cycle. Accumulating losses over a full cycle produces a successful certificate even with a suboptimal witness. Calibration ties those losses to feasible welfare; positive margins quantify the decision errors and payoff changes the conclusion can withstand. Under compactness and attainment, the converse identifies when strict certificates exist and the supremal margins. Incomplete welfare information can therefore establish an exact conclusion about every optimizer's eventual behavior; the converse identifies the best margins available within the certificate class.

A Foundations and classification

A.1 Foundations and the welfare bound

Maintain (2.1): labor lies in $[0, 1]$, the primitive vector is $p = (A, \beta, q, \eta, \theta, \rho, w)$, and $1 < \theta < 2$. Utility is $-\infty$ at $C = 0$ or $L = 1$. With the capacity coefficient H_a in (2.4), put

$$R_p = q + AH_a, \quad a_p = \frac{\eta}{1 - \beta}, \quad \mathcal{D}_p = \frac{\eta}{(1 - \beta)^2} \log \frac{R_p}{q}. \quad (\text{A.1})$$

These constants give a finite welfare bound from primitives alone.

Lemma 1 (Economic foundations). *At every primitive vector in (2.1), value is finite and continuous on $(0, \infty)$, and optimal histories exist. The optimal history and successor correspondences have compact values and closed graphs locally on that domain. Bellman's equation holds, and every optimal tail is optimal. Every optimal action exhausts resources and has positive consumption, positive successor capital and $L < 1$. On compact positive initial stock intervals, optimal first consumption, successor and leisure are uniformly bounded away from zero. If optimal stocks converge to $k > 0$, then $k \in G_p(k)$.*

Proof. Output is at most $P = A(1 - w)^{\theta/\rho}$. Every feasible history from K stays below $\max\{K, P/(1 - q)\}$, and a common stock bound works on compact initial stock intervals. Retaining consumption as a coordinate makes the feasible history set closed in a common compact product of intervals. Extend utility upper semicontinuously at zero consumption and zero leisure. If M_u bounds current utility above on this set, welfare is $M_u/(1 - \beta)$ minus the sum of the nonnegative discounted losses $M_u - u_p$. Fatou's lemma therefore gives upper semicontinuity, including at welfare $-\infty$. Cake has finite welfare (4.1), so the maximum is finite and attained. Zero consumption, labor one, or a zero successor forcing zero consumption later gives welfare $-\infty$. Increasing first consumption shows that disposal cannot be optimal.

Take limits of optimizers on a common compact cover to obtain upper semicontinuity of value. For lower semicontinuity, retain an optimizer's first labor, successor and entire tail. Its positive first consumption absorbs small changes in initial resources. Since its labor is strictly below

one, the resulting welfare varies continuously. Value is therefore continuous. Compactness also gives closed optimal history graphs and compact optimal sets; their projections give the successor claims. Over a compact positive initial stock interval, the graph of optimal histories is compact in a common product cover. Its first consumption, successor and leisure are positive continuous coordinates, so their minima are positive. Splitting and concatenating optimal histories proves Bellman's equation and optimality of tails. Finally, closedness of the optimal successor graph applied to $(K_t, K_{t+1}) \rightarrow (k, k)$ gives $k \in G_p(k)$. $\square$

Lemma 2 (Cake normalization and its terminal term). *On positive consumption and successor edges define $r_p^0 = u_p - E_p(K) + \beta E_p(K')$. For a history π from K_0 , write $J_{K_0}^0(\pi) = \sum_{t \geq 0} \beta^t r_p^0(e_t)$ for normalized welfare. Original and normalized welfare are real on the same positive histories. On that class, $\beta^t E_p(K_t) \rightarrow 0$ and $J_{K_0}^0(\pi) = J_{K_0}(\pi) - E_p(K_0)$. An edge with $L = 1$ has reward $-\infty$ in both formulations.*

Proof. The capacity bound gives $F_p(K, L)/K \leq AH_a L^{\theta-1}$ and resources at most $R_p K$. Put $s = K'$ for successor capital. For positive C, s , write $Q = qK + F_p(K, L)$ and $z = s/Q$. Direct subtraction gives

$$\begin{aligned} u_p(C, L) - E_p(K) + \beta E_p(s) &= \frac{\eta}{1-\beta} \left\{ \log \frac{Q}{qK} - D_\beta(z) \right\} + (1-\eta) \log(1-L) \\ &\quad + \eta \log \frac{C}{Q-s}, \\ D_\beta(z) &= (1-\beta) \log \frac{1-\beta}{1-z} + \beta \log \frac{\beta}{z} \geq 0. \end{aligned} \tag{A.2}$$

The last inequality follows by differentiation; equality holds at $z = \beta$. The identity bounds every normalized reward above by $a_p \log(R_p/q)$. Original rewards are bounded above on the history's resource cover. Both discounted sums are therefore well defined.

For a history with finite discounted welfare, bound $\log C_t$ above by M_C on its resource cover. The losses $\eta(M_C - \log C_t) + (1-\eta)[- \log(1-L_t)]$ form a summable nonnegative discounted series. Each component is nonnegative, so $\beta^t \log C_t \rightarrow 0$. The inequality $\log C_t - \log R_p \leq \log K_t$, together with the finite upper stock cover, gives $\beta^t E_p(K_t) \rightarrow 0$. For each finite positive prefix,

$$\sum_{t < T} \beta^t r_p^0(e_t) = \sum_{t < T} \beta^t u_p(e_t) - E_p(K_0) + \beta^T E_p(K_T). \tag{A.3}$$

This proves the welfare identity when original welfare is real. If original welfare is $-\infty$, its partial sums tend to $-\infty$, while $E_p(K_T)$ is bounded above on the stock cover. The normalized sum is then $-\infty$ too. Original welfare cannot be $+\infty$, so a real normalized sum implies real original welfare. Thus the finite welfare classes coincide without subtracting infinite payoffs. A labor one edge makes both sums $-\infty$. $\square$

Proof of Proposition 4. The preceding identity gives the uniform stage bound $r_p^0 \leq a_p \log(R_p/q)$. For histories with finite welfare, the terminal identity therefore yields $J_{K_0}(\pi) - E_p(K_0) \leq \mathcal{D}_p$. The bound also holds at welfare $-\infty$, including zero consumption, labor one and zero successors. Cake gives the lower bound. Thus

$$0 \leq V_p(K) - E_p(K) \leq \mathcal{D}_p.$$

Lemma 1 supplies attainment and optimal tails. Replacing an optimal tail at date n by cake gives exactly (4.3); its loss is at most $\beta^n \mathcal{D}_p$. Letting $n \rightarrow \infty$ proves equality of the extinct and unrestricted welfare suprema.

The same bound keeps the relevant Bellman actions away from the utility boundaries. On a compact positive initial stock interval, every Bellman action whose objective is bounded below by a common finite constant has consumption, successor and leisure bounded away from zero. Current resources and continuation welfare have common upper bounds. Zero consumption or zero leisure would therefore force the objective to $-\infty$. A successor approaching zero does the same because $V_p(s) \leq E_p(s) + \mathcal{D}_p \rightarrow -\infty$. The argument applies to both restricted and unrestricted Bellman problems. $\square$

A.2 Optimal action bounds and productive quantities

The resource constant $R_p = q + AH_a$ and the welfare bound $\mathcal{D}_p$ in (4.2) define

$$\chi_p = (1 - \beta)q \left(\frac{\beta q}{R_p} \right)^{\beta/(1-\beta)} \exp(-\beta \mathcal{D}_p/\eta), \quad 0 < \chi_p < q. \quad (\text{A.4})$$

Proposition 7 (Consumption, leisure and productive activity). *At every primitive vector in (2.1) and every positive stock K , each optimal action obeys*

$$C(1 - L)^\nu \geq \chi_p K, \quad C \geq \chi_p K, \quad 1 - L \geq (\chi_p/R_p)^{1/\nu} > 0. \quad (\text{A.5})$$

If an optimal tail remains in $[h, \bar{K}]$, where $0 < h \leq \bar{K} < \infty$, put $c = \chi_p h$, $d_0 = (q - \chi_p)(\bar{K} - h)$ and $f_0 = (1 - q)h + c$. Consumption on that tail is at least $c > 0$, and every window of $T \geq 1$ dates satisfies

$$\frac{1}{T} \sum_{j=0}^{T-1} F_p(K_{t+j}, L_{t+j}) \geq f_0 - \frac{d_0}{T}. \quad (\text{A.6})$$

Proof of Proposition 7 and Corollary 1. For an optimal action with positive successor s , Bellman's equation and the independently established cake bounds give

$$\begin{aligned} \eta \log[C(1 - L)^\nu] &= V_p(K) - \beta V_p(s) \\ &\geq E_p(K) - \beta E_p(s) - \beta \mathcal{D}_p \\ &\geq E_p(K) - \beta E_p(R_p K) - \beta \mathcal{D}_p. \end{aligned}$$

The last inequality uses $s \leq R_p K$ and the fact that cake value increases with capital. Expanding (4.1) and exponentiating gives $C(1 - L)^\nu \geq \chi_p K$. Since $1 - L \leq 1$, this implies $C \geq \chi_p K$; since $C \leq R_p K$, it also gives the leisure floor. Formula (A.4) gives $0 < \chi_p < q$. These restrictions apply to optimal actions only; the barrier comparisons still cover the complete feasible graph.

On an optimal tail in $[h, \bar{K}]$, resource exhaustion gives

$$\begin{aligned} \sum_{j=0}^{T-1} F_p(K_{t+j}, L_{t+j}) &\geq (\chi_p - q)K_t + (1 - q + \chi_p) \sum_{j=1}^{T-1} K_{t+j} + K_{t+T} \\ &\geq (1 - q + \chi_p)hT - (q - \chi_p)(\bar{K} - h). \end{aligned}$$

This proves the window bound, including $T = 1$.

Finally every optimal transition satisfies $K' \leq R_p K - C \leq (R_p - \chi_p)K$. If $A \leq A_{\text{rep}}$, then $R_p \leq 1$ and $0 < q - \chi_p < R_p - \chi_p < 1$. Thus the explicit contraction factor is

$$\gamma_p = R_p - \chi_p \in (0, 1). \quad (\text{A.7})$$

Iteration proves (2.9), independently of classification or productive capture. $\square$

A.3 Ordering the physical and Euler boundaries

The orderings in Figure 3 follow directly from the two peaks in Proposition 1. On positive inputs, $F_{p,KL} > 0$, because $\theta - \rho > 0$. Thus there exists a labor choice strictly between zero and one satisfying $\beta[q + F_{p,K}] > 1$ if and only if $Ay'(K) > D$. The strict unimodality and vanishing endpoint limits of y' give the empty set for $A \leq A_{\text{ret}}$ and the two simple roots $K_E^- < k_\Phi < K_E^+$ for $A > A_{\text{ret}}$. Also $k_\Phi < k_a$. At the upper physical root,

$$Ay'(K_p^+) = \delta\theta s(K_p^+) < \delta < D.$$

Since $K_p^+ > k_a > k_\Phi$, this root lies strictly to the right of K_E^+ . The only strict orderings when both intervals exist are therefore

$$\begin{aligned} K_E^- < K_E^+ < \kappa_p < K_p^+, \quad K_E^- < \kappa_p < K_E^+ < K_p^+, \\ \kappa_p < K_E^- < K_E^+ < K_p^+. \end{aligned}$$

The primitive conditions identify when each case occurs. Put

$$d = \frac{D}{\delta} > 1, \quad r_* = \frac{\Phi_a}{H_a}, \quad d_c = \frac{\theta(1-\rho)}{\theta-\rho}. \quad (\text{A.8})$$

Since $y'(k_a) = H_a < \Phi_a$ and $\Phi_a = d_c y(k_\Phi)/k_\Phi < d_c H_a$, one has $1 < r_* < d_c < \theta$. Moreover $A_{\text{ret}}/A_{\text{rep}} = d/r_*$. Thus favorable returns appear first when $d < r_*$, maintenance appears first when $d > r_*$, and the two productivity thresholds coincide when $d = r_*$. These conditions give panels (a)–(c), together with their equality cases.

For $1 < d < \theta$, the unique stock satisfying $\theta s(K) = d$ and its replacement productivity are

$$k_d = \left[\frac{d(1-w)}{w(\theta-d)} \right]^{1/\rho}, \quad A_{\text{cross}} = \frac{\delta k_d}{y(k_d)}. \quad (\text{A.9})$$

Here $k_d < k_a$, so $A_{\text{cross}} > A_{\text{rep}}$, and $A_{\text{cross}} \geq A_{\text{ret}}$, with equality only when $k_d = k_\Phi$, or $d = d_c$. At a common physical and Euler root, division of the two root equations requires $\theta s(K) = d$; hence this is the only possible crossing.

If $1 < d < d_c$, then $k_d > k_\Phi$. For $\max\{A_{\text{rep}}, A_{\text{ret}}\} < A < A_{\text{cross}}$, panel (d) applies. At $A = A_{\text{cross}}$, $K_E^+ = \kappa_p$; above it, panel (e) applies. If $d_c < d < \theta$, then $k_d < k_\Phi$. Panel (f) applies for $A_{\text{ret}} < A < A_{\text{cross}}$, the lower roots coincide at A_{cross} , and panel (e) applies above it. These directions follow because κ_p decreases from k_a to zero as A rises, while $Ay'(\kappa_p) = \delta\theta s(\kappa_p)$ strictly increases from δ to $\theta\delta$. At $d = d_c$, the Euler double root equals κ_p at $A = A_{\text{ret}}$, and panel (e) applies for every larger A .

Finally, if $d \geq \theta$, then $Ay'(K) > D$ and $Ky'(K)/y(K) < \theta$ imply $Ay(K)/K > \delta$. The return interval is therefore strictly inside the physical interval, as in panel (f). At $A = A_{\text{rep}}$, the same upper-root argument at k_a gives $K_E^+ < k_a$ whenever the return interval exists. When $A_{\text{ret}} > A_{\text{rep}}$,

the equality point k_Φ at $A = A_{\text{ret}}$ lies below κ_p , at κ_p , or inside the physical interval according as $d < d_c$, $d = d_c$, or $d > d_c$. If $A_{\text{ret}} = A_{\text{rep}}$, its equality point k_Φ lies strictly below the physical singleton k_a . This accounts for the boundary cases as well.

A.4 Physical boundaries under productivity losses

Proof of Proposition 2. At either simple replacement root, $\log A + \log y(K) - \log K = \log \delta$. Differentiating and using $Ky'(K)/y(K) = \theta s(K)$ gives (2.7). The denominator is positive below k_a and negative above it. This proves the elasticity signs and the nesting of the intervals. As $A \downarrow A_{\text{rep}}$, the roots converge to the unique maximum k_a of $y(K)/K$. At that stock $\theta s(k_a) = 1$, so both elasticities diverge in absolute value.

At productivity $A_e > A_{\text{rep}}$, write $\Phi_e(K) = qK + A_e y(K)$. Below $\kappa(A_e)$ one has $\Phi_e(K) < K$. On any $[0, K_1]$ with $0 < K_1 < \kappa(A_e)$, the ratio $\Phi_e(K)/K$, extended by q at zero, has a maximum strictly below one. Every feasible history starting there contracts to zero. At $\kappa(A_e)$, positive first consumption sends the successor strictly below the root. Every finite welfare history has positive consumption, so extinction follows throughout the closed band in the statement. Every stock in that band lies strictly inside the original sustainable interval.

The identity for τ^* follows from $A_{\text{rep}}/A = \lambda_{\text{rep}}^{\theta-1}$. When $\tau > \tau^*$, effective productivity is strictly below the replacement threshold, and Proposition 1 gives the global feasible contraction bound. At equality, optimal contraction follows instead from Corollary 1. $\square$

Example 1 (Closed form physical boundaries). The harmonic technology $F(K, L) = A[2KL/(K + L)]^{3/2}$ belongs to the admitted CES family. Put $z = 8A^2/\delta^2$. Its sustainable set is empty for $z < 27/4$ and is $\{1/2\}$ at equality. For $z > 27/4$, let $\vartheta = \frac{1}{3} \arccos[-3\sqrt{3}/(2\sqrt{z})]$. Then

$$\begin{aligned}\kappa(A) &= 2\sqrt{z/3} \cos(\vartheta - 2\pi/3) - 1, \\ K^+(A) &= 2\sqrt{z/3} \cos \vartheta - 1.\end{aligned}\tag{A.10}$$

Indeed, replacement gives $(K + 1)^3 = zK$. Substituting $K + 1 = 2\sqrt{z/3} \cos v$ gives $\cos(3v) = -3\sqrt{3}/(2\sqrt{z})$. The displayed branches are the two positive roots; the third is negative. These formulas locate physical maintenance. Optimal preservation requires the welfare comparison in Theorem 1.

A.5 Local order and the frontier

Lemma 3 (An ordered frontier between two anchors). *Suppose the conclusions of Lemma 1 hold. Let $0 < a < b < h$. Suppose every optimum from $(0, a]$ is extinct, while every optimum from $[b, \infty)$ enters and remains in a fixed compact target contained in $[h, \infty)$. Suppose also that*

$$\max G(x) \leq \min G(y) \quad (a \leq x < y \leq b), \quad x \notin G(x) \quad (a \leq x \leq b).$$

There is a unique $S \in (a, b)$ with unanimous extinction below, unanimous productive survival above, and attained coexistence at S . Within the bracket, extinct successors are strictly below their current stock, productive successors strictly above it, and the optimal successors from S lie on both sides and never equal S .

Proof. Every finite chain of Bellman maximizing actions extends to an optimum by appending an optimal tail. For an infinite maximizing chain, extend each finite prefix in this way. Compactness

of the optimal history set at the fixed initial stock gives a convergent subsequence of these optima. Every fixed finite prefix eventually agrees with the prescribed chain, so the limit is that chain and is optimal. In particular, least and greatest successor selections generate optima. This argument needs no additional terminal limit for the chain.

Direction and finite exit. Suppose $x < y \in G(x)$ and both stocks are in $[a, b]$. Order gives $y \leq \min G(y)$, and exclusion of fixed optimal transitions makes the inequality strict. A downward move likewise forces all subsequent moves downward while the history remains in the bracket. Any optimal history confined to $[a, b]$ would therefore be strictly monotone and converge to $k \in [a, b]$. Closedness applied to successive stocks would imply $k \in G(k)$, a contradiction. Every optimum exits the bracket in finite time, toward the anchor corresponding to its first move. An upward optimum from a would reach the productive anchor, contradicting extinction there. A downward optimum from b would reach the extinct anchor, contradicting productive capture. Thus all successors at a are below a , and all at b are above b .

An upward history cannot start below a downward history. Suppose an upward optimum starts at x_0 below a downward optimum starting at y_0 , both inside the bracket. As long as $x_t < y_t$, order gives $x_{t+1} \leq y_{t+1}$. Equality at a stock z is impossible: direction gives $x_t < z < y_t$, while order between these three origins forces $G(z) = \{z\}$. Thus the inequality remains strict. Both histories are trapped in $[x_0, y_0]$: the lower one increases and cannot cross the decreasing upper one. Neither can exit the bracket first, since an upward exit would cross the upper history and a downward exit would cross the lower history. This contradicts finite exit.

The frontier and attainment. Consequently the states with only downward successors form a down set, and those with only upward successors form an up set. They are nonempty at a and b and relatively open by upper hemicontinuity. Every other state has successors on both sides, since its compact successor set omits the state itself. There is at most one such state: two would give an upward optimum below a downward optimum. The two disjoint, nonempty, relatively open sets cannot cover the connected interval $[a, b]$. Hence there is a remaining state, denoted $S \in (a, b)$, and the preceding argument makes it unique. Every downward choice at S , followed by an optimal tail, yields an attained extinct optimum; every upward choice yields an attained productive optimum. The anchors and direction argument classify each tail, and the two classes exhaust the optimum set. $\square$

Primitive verification constants

We now impose the patient condition (2.3) for the classification proof. The construction first bounds possible stationary optima, then chooses the regions for productive capture. The stock b marks the start of certified capture, h is the target's lower boundary, and $\bar{K}(A)$ bounds the invariant resource cover.

Stationary bounds. Put $\alpha = 1/\nu$, $\xi = -1/\rho$, $c_w = (1 - w)/w$, and

$$z_C = \frac{(\theta - 1)\xi}{1 + \xi}, \quad L_* = \frac{\theta}{\theta + \nu}.$$

The stationary candidates will be parameterized by $z > z_m := -E/D > 0$:

$$\begin{aligned} L(z) &= \frac{\theta D z}{\nu E + (\theta + \nu) D z}, & K(z) &= c_w^{-\xi} z^\xi L(z), \\ \mathcal{A}(z) &= \frac{D(1 - w)^{\theta\xi} c_w^{-\xi} (1 + z)^{1+\theta\xi}}{\theta L(z)^{\theta-1} z^{(\theta-1)\xi}}. \end{aligned} \tag{A.11}$$

Here $L_* < L(z) < 1$. Put $z_u = \max\{z_m, z_C\}$ and

$$k_* = c_w^{-\xi} L_* z_u^\xi, \quad A_s = \frac{D(1-w)^{\theta\xi} c_w^{-\xi} (1+z_u)^{1+\theta\xi}}{\theta L_*^{\theta-1} z_m^{(\theta-1)\xi}}. \quad (\text{A.12})$$

Witness and resource bounds. Fix witness labor $\sigma = 3/4$ while retaining the full feasible labor domain. Distinguish witness output from physical capacity:

$$f(K) = \{wK^\rho + (1-w)\sigma^\rho\}^{\theta/\rho}, \quad f_1(K) = \{wK^\rho + 1-w\}^{\theta/\rho}, \\ M_0 = (1-w)^{\theta/\rho}, \quad \gamma > M_0/\delta, \quad Q = q\gamma + M_0.$$

Target regions. Choose the cover coefficient first; $\gamma = 1 + 2M_0/\delta$ is admissible. Define

$$h_* = \min\{1/2, k_*/2, 1/9, 2k_a\}, \\ T = \exp \max \left\{ \log \frac{f(1)}{f(h_*)}, \right. \\ \left. \frac{1 - \nu \log(1-\sigma) - (1-\beta^2)\theta \log \sigma + \beta^2 \log[Q/f(1)]}{\beta(1-\beta)} \right\}, \quad (\text{A.13}) \\ h = \left[\frac{(f(1)/T)^{\rho/\theta} - (1-w)\sigma^\rho}{w} \right]^{1/\rho}, \\ b = h/2, \quad \bar{K}(A) = \gamma A, \quad t = \frac{e^{1/2} - 1}{e^{1/2} + 1}.$$

Consequently $0 < b < h \leq h_* < 1$, $h < k_*$, $9b < 1$, and $b \leq k_a$. Since $\bar{K}(A) = \gamma A$, one has $R_{\text{cap}} = AQ$.

Productivity gates. The productivity threshold below has five terms. In order, they ensure witness feasibility and the strict welfare margin; exclusion of stationary optima below the target; maintenance at $b/2$, which places the lower physical root below the capture region; positivity of the reduced return's cross derivative; and global domination of zero labor. The last two terms establish local successor order over the full labor choice set. Put

$$j(u) = \theta - 1 - (\theta - \rho) \frac{w}{w + (1-w)u^\rho}, \\ t_\varphi = \left[\frac{w(1-\rho)}{(1-w)(\theta-1)} \right]^{1/\rho}, \quad j_* = j(t_\varphi/2) > 0, \\ c = \min \left\{ t_\varphi/2, \frac{\alpha j_*}{b[2(1+\alpha) + \alpha j_*]} \right\}, \\ f_o(u) = \frac{\theta w}{u} \{w + (1-w)u^\rho\}^{\theta/\rho-1}, \quad (\text{A.14}) \\ A_* = \max \left\{ \frac{1}{t f(b)}, A_s, \frac{\delta b}{2 f_1(b/2)}, \right. \\ \left. \frac{q b^{2-\theta}}{\alpha f_o(c)}, \frac{q b^{1-\theta} [(1-9b)^{-\nu} - 1]}{[w + (1-w)9^\rho]^{\theta/\rho}} \right\}.$$

Once the cover coefficient is fixed, these finite constants depend only on the six primitives other than A . Fix $A > A_*$ and write $Z(K) = qK + Af(K)$ and $\Phi(K) = qK + Af_1(K)$.

Lemma 4 (Productive anchor and capture). *Under (A.13)–(A.14), every optimal successor from $K \geq b$ is at least h . Every optimum from that domain eventually enters $[h, \bar{K}(A)]$ and remains there.*

Proof. Witness and unrestricted ceiling. Output at full labor is below AM_0 , so $[0, \bar{K}(A)]$ is invariant and $\Phi(\bar{K}(A)) < AQ < \bar{K}(A)$. Also $tf(b) < M_0 < \gamma$ and $A > 1/[tf(b)]$ give $1 < \bar{K}(A)$. Labor σ , resetting capital to one and then maintaining it there, earns (4.8). For $K \geq b$, one has $Z(K) > 1/t$ and hence $Z(K) - 1 > (1 - t)Z(K)$. Consumption after the reset exceeds $(1 - t)Af(1)$, so the witness is feasible throughout the domain. Every unrestricted continuation from $0 < y \leq \bar{K}(A)$ has the independent ceiling (4.9): first consumption is at most $\Phi(y)$, and invariance bounds every later consumption by AQ . This ceiling uses the full resource bound and nonpositive leisure utility; it allows recovery after a low successor.

Excluding low successors. Homogeneity and monotonicity give $f(K) = \sigma^\theta f_1(K/\sigma) \geq \sigma^\theta f_1(K)$. Hence $\Phi(K) \leq \sigma^{-\theta} Z(K)$ at every positive stock. Moreover $Z(h) \leq (1 + t)Af(h)$, since $h < 1$ and $1 < tAf(b)$. For $K \geq b$ and $0 < y \leq h$, current reward and the unrestricted continuation therefore have the bound

$$u_p(C, L) + \beta \bar{V}(y) \leq \eta \log Z(K) + \beta \eta \log Z(h) - (1 + \beta)\eta \theta \log \sigma + \frac{\beta^2 \eta}{1 - \beta} \log(AQ).$$

The bound covers every $L \in [0, 1]$, with value $-\infty$ at full labor. The advantage of $W(K)$, multiplied by $(1 - \beta)/\eta$, is at least

$$\begin{aligned} & \beta(1 - \beta) \log \frac{f(1)}{f(h)} - \beta^2 \log \frac{Q}{f(1)} + \nu \log(1 - \sigma) + (1 - \beta^2)\theta \log \sigma \\ & + \log(1 - t) - \beta(1 - \beta) \log(1 + t) \geq \frac{1}{2}. \end{aligned}$$

The first line is at least one by (A.13); the last two terms are at least $\log[(1 - t)/(1 + t)] = -1/2$.

Immediate entry and capture. A zero successor has value $-\infty$. The strict comparison just proved therefore excludes every optimal successor at or below h from all $K \geq b$, including stocks above the cover. Optimal tails imply permanence above h : every later stock remains in the domain of the same comparison. On the invariant cover, entry from M is immediate. For larger initial stocks, the resource bound gives $K_n \leq q^n(K_0 - AM_0/\delta) + AM_0/\delta$. Since $\bar{K}(A) > AM_0/\delta$, every such history enters the cover in finite time and then remains there. This proves the lemma.

Verification of the full certificate. The same inputs also yield the FPBB displayed in Subsection 4.3. For its staying comparison, introduce the auxiliary bound

$$\hat{B}(K) = \eta \log \Phi(K) + \beta \eta \log \Phi(h) + \frac{\beta^2 \eta}{1 - \beta} \log(AQ).$$

The upper bound for a low successor also bounds $\hat{B}$, since $\Phi \leq \sigma^{-\theta} Z$. Thus $W - \hat{B} \geq \varepsilon_0$ on M . Moreover

$$\hat{B} - B_M = \frac{\lambda}{1 - \beta}, \quad \lambda = \beta^2 \eta \log \frac{AQ}{\Phi(h)} > 0.$$

This proves the larger staying comparison in (4.11). Every adverse action with its unrestricted continuation still loses at least ε_0 to W . The direct staying inequality displayed after (4.10) covers

every staying action, including those of reward $-\infty$. Boundedness on $M = [b, h]$ gives the terminal condition. The witness resets to $1 \in H$ and remains there. Invariance excludes successors above $\bar{K}(A)$, and the low successor bound covers both adverse classes, including recovery. Thus every hypothesis of Theorem 3 holds. In this application the stronger exclusion of all low successors already gives immediate entry; the staying barrier verifies the complete certificate without requiring a longer block construction. $\square$

Lemma 5 (Global labor choice and local successor order). *Under the same construction, for $0 < K \leq b$, labor maximization at any successor permitting positive consumption has a unique solution in $(0, 1)$. The globally maximized return has strict increasing differences on feasible crossed rectangles in this stock domain, and $\max G_p(x) \leq \min G_p(y)$ for $0 < x < y \leq b$.*

Proof. For $0 < K \leq b$, replace any feasible zero labor choice by $9K$, keeping its successor fixed. The gain is at least

$$J(K) = \eta \log(1 + d_9 K^{\theta-1}) + (1 - \eta) \log(1 - 9K), \quad d_9 = \frac{A}{q} [w + (1 - w)9^\rho]^{\theta/\rho}.$$

Its derivative has the sign of

$$\eta d_9 (\theta - 1) - 9(1 - \eta) K^{2-\theta} - 9d_9 [1 - \eta + \eta(\theta - 1)] K,$$

a strictly decreasing expression. Thus the gain can first rise and then fall, but cannot have an interior minimum. The last gate in (A.14) gives $J(b) > 0$; $J(0) = 0$ then implies $J(K) > 0$ on $(0, b]$. Thus zero labor is either infeasible or strictly suboptimal. Labor one has utility $-\infty$.

At fixed K , define $\zeta(L) = F_p - \alpha(1 - L)F_{p,L}$ and $T_L = (1 + \alpha)F_{p,L} - \alpha(1 - L)F_{p,LL}$. For positive consumption, the labor derivative at fixed successor is

$$\frac{\partial}{\partial L} u_p(qK + F_p(K, L) - s, L) = \frac{1 - \eta}{C(1 - L)} [s - qK - \zeta(L)].$$

Thus the interior labor equation is $s - qK = \zeta(L)$. The sign of $T_L = \zeta'(L)$ is the sign of $(1 + \alpha)L/(1 - L) - \alpha j(L/K)$, a strictly increasing expression. It is negative near zero and tends to infinity as $L \uparrow 1$. Thus ζ decreases and then increases. A root on its decreasing branch changes the utility derivative from negative to positive and can only be a local minimum. Its increasing branch has at most one root, where the derivative changes from positive to negative. At the turning point itself, a tangency does not create a maximum. The feasible positive consumption labor set is an interval; utility tends to $-\infty$ at a positive lower consumption boundary and at labor one. Since zero labor is infeasible or dominated, the global maximum is unique, interior, and has $T_L > 0$. The choice of c gives

$$\frac{(1 + \alpha)cb}{1 - cb} \leq \frac{\alpha j_*}{2} < \alpha j(c),$$

so every maximizing labor exceeds cK .

Let $r_p(K, s)$ denote this labor maximized return. Implicit differentiation on its maximizing branch first gives

$$r_{p,Ks} = \frac{\eta\alpha}{C^2 T_L} [(q + F_{p,K})\{F_{p,L} - (1 - L)F_{p,LL}\} + (1 - L)F_{p,L}F_{p,KL}].$$

Substituting the CES derivatives reduces this expression to

$$\begin{aligned} r_{p,Ks} &= \frac{\eta\alpha\mathcal{N}}{C^2T_L}, & \mathcal{N} &= \frac{F_{p,L}}{L}\{qn + F_{p,K}[1 - \rho(1 - L)]\}, \\ n &= L - (1 - L)j(L/K) > -L/\alpha. \end{aligned} \quad (\text{A.15})$$

If $n \geq 0$, this is positive. If $n < 0$, then $j(L/K) > 0$. On the interval where $j(u) > 0$, $f_o(u)$ increases: its logarithmic derivative times u is $(\theta - \rho)[1 - w/(w + (1 - w)u^\rho)] - 1 > -\rho > 0$. The function j decreases, so its positive set is an interval containing both c and L/K . Since $L/K > c$, the fourth gate gives $F_{p,K}/L = AK^{\theta-2}f_o(L/K) > q/\alpha$. Consequently $\mathcal{N} > 0$ again. Integrating across a feasible crossed rectangle gives strict increasing differences.

Suppose $0 < x < y \leq b$ had optimal successors $s_x > s_y$. Since s_x is feasible with positive consumption at x , every successor in $[s_y, s_x]$ is feasible at every stock in $[x, y]$: resources increase with capital and a smaller successor releases consumption. Bellman optimality would give

$$\begin{aligned} r_p(x, s_x) + \beta V_p(s_x) &\geq r_p(x, s_y) + \beta V_p(s_y), \\ r_p(y, s_y) + \beta V_p(s_y) &\geq r_p(y, s_x) + \beta V_p(s_x). \end{aligned}$$

Cancelling the continuation values contradicts strict increasing differences on that rectangle. Therefore

$$\max G_p(x) \leq \min G_p(y) \quad (0 < x < y \leq b). \quad (\text{A.16})$$

□

Lemma 6 (Exclusion of low stationary optima). *Under the same construction, $K \notin G_p(K)$ for every $0 < K \leq b$.*

Proof. Bellman's equation allows a fixed optimal transition to be repeated at a stationary action (C, L, K) . Keep labor at its stationary level and vary the date one stock to $K + v$, restoring capital to K at date two. Consumption becomes $C - v$ initially and $q(K + v) + F_p(K + v, L) - K$ one date later. Both signs of sufficiently small v are feasible. The derivative of welfare at zero is

$$-\frac{\eta}{C} + \beta \frac{\eta}{C}(q + F_{p,K}) = 0,$$

so $F_{p,K} = D$. This necessary condition uses only a feasible variation over two dates; no concavity or Euler sufficiency is used.

Positive consumption requires $L > 0$, and finite welfare requires $L < 1$, so the labor condition is interior. Put $z = c_w(K/L)^{-\rho}$. The factor identities are

$$KF_{p,K} = \frac{\theta F_p}{1 + z}, \quad LF_{p,L} = \frac{\theta F_p z}{1 + z}.$$

Using $F_{p,K} = D$ and $C = F_p - \delta K$ gives $C/K = (E + Dz)/\theta$. Thus $z > z_m$. Substitution into $\nu C = (1 - L)F_{p,L}$ gives $L = \theta Dz/[\nu E + (\theta + \nu)Dz]$. The definition of z then gives $K(z)$, and the production identity gives $\mathcal{A}(z) = A$ in (A.11).

Since $E < 0$, every candidate has $L(z) > L_*$. On $(z_m, z_u]$, (A.12) bounds $\mathcal{A}(z)$ above by A_s , excluding every root when $A > A_s$. For $z > z_u$, $K(z) > k_* > h$. Thus no fixed optimal transition exists at $0 < K \leq b$. □

Proof of Theorem 1. Physical extinction and the frontier. Full capacity average product $f_1(K)/K$ increases until k_a and then decreases to zero. The third gate gives $Af_1(b/2) > \delta b/2$; since $b \leq k_a$,

there are two physical replacement roots and the lower one satisfies $\kappa_p < b/2$. Below that root, $\Phi(K) < K$. Positive first consumption at the root sends capital strictly below it. Once a finite welfare history reaches $0 < K_1 < \kappa_p$, the continuous ratio $\Phi(K)/K$, extended by $q < 1$ at zero, has a maximum strictly below one on $[0, K_1]$. Such a history contracts to zero. Every optimum from $K \leq \kappa_p$ is therefore extinct.

Apply Lemma 3 with lower anchor κ_p , upper anchor b and the productive target already established. It gives the unique frontier, attained coexistence and successor directions. An extinct optimum from the frontier first moves below it. Order forces strict decreases until the physical extinction region, where the resource constraint does so thereafter. Proposition 1 gives constant maintenance with positive consumption and leisure on $(\kappa_p, b]$.

Primitive neighborhoods and frontier continuity. The inequalities establishing patience, the five verification gates and the exclusion of stationary optima are strict. Their bounds depend continuously on the six primitives other than A . The lower physical root is simple. Classification therefore persists on an open neighborhood in all seven primitives, with a common resource cover after shrinking the neighborhood.

For joint continuity of the optimal graph, take a compact primitive neighborhood and a common invariant cover $[0, \bar{K}]$. Write $K = \bar{K}k$ and

$$g_p(k, L) = q + \frac{F_p(\bar{K}k, L)}{\bar{K}k} \quad (k > 0), \quad g_p(0, L) = q.$$

The bound $0 \leq g_p - q \leq Aw^{\theta/\rho}(\bar{K}k)^{\theta-1}$ gives joint continuity at zero, uniformly in labor and primitives on that neighborhood. The normalized Bellman operator is

$$(\mathcal{T}_p\varphi)(k) = \sup_{0 \leq L \leq 1, 0 < z < 1} \left\{ \frac{\eta}{1-\beta} [\log(g_p/q) - D_\beta(z)] + (1-\eta) \log(1-L) + \beta\varphi(kg_pz) \right\}.$$

At labor one the objective is $-\infty$. The cover gives $kg_p \leq 1$, so the displayed controls are feasible. Choose a common bound $\bar{\mathcal{D}} \geq \mathcal{D}_p$ on the primitive neighborhood. If $0 \leq \varphi \leq \bar{\mathcal{D}}$, the stage bound gives $\mathcal{T}_p\varphi \leq (1-\beta)\mathcal{D}_p + \beta\bar{\mathcal{D}} \leq \bar{\mathcal{D}}$. The action $L = 0, z = \beta$ has zero normalized current reward and gives a nonnegative incumbent. Thus every operator preserves the same function interval $[0, \bar{\mathcal{D}}]$, including all iterates starting from zero. On the compact primitive neighborhood, η , $1-\eta$, β and $1-\beta$ have positive lower bounds. Consequently the divergence of $D_\beta(z)$ at either endpoint excludes z near zero or one uniformly from every maximizer; the bounded continuation term cannot offset it. The same argument with $\log(1-L)$ excludes labor near one. Each maximization therefore uses a fixed compact control rectangle and a jointly continuous objective.

The operators satisfy $\|\mathcal{T}_p\varphi - \mathcal{T}_p\psi\|_\infty \leq \beta_{\max}\|\varphi - \psi\|_\infty$, with $\beta_{\max} < 1$. Their continuous iterates converge uniformly in states and primitives. At $k = 0$, the resource gain vanishes and cake attains the maximum zero, so every iterate is zero there. At positive stocks, Bellman's equation and normalization make $V_p(\bar{K}k) - E_p(\bar{K}k)$ a bounded fixed point. Uniqueness identifies it with the limit, proving joint continuity at positive stocks. The uniform first action bounds and Bellman's equation then give a jointly closed optimal successor graph and joint upper hemicontinuity.

Choose $x < S_p < y$ arbitrarily close to the frontier, inside its ordered interval. The strict directions $\max G_p(x) < x$ and $\min G_p(y) > y$ persist under small primitive changes. Continuity of the physical root and verification stocks keeps x, y inside the nearby ordered intervals. These directions place every nearby frontier between x and y . Their arbitrary proximity to the original frontier proves continuity. $\square$

B Frontier geometry and allocation

Lemma 7 (Separation of active capital slopes). *At a classified frontier, the extinct and productive optimal history sets are compact and their first successors are strictly separated. Every productive active capital slope exceeds every extinct one.*

Proof. Optimal histories form a compact set by Lemma 1. Classification excludes the current stock from its optimal successors and splits the histories into two closed sets, with successors strictly below and above the frontier. Unique global labor maximization in Lemma 5 identifies $a_\pi = r_{p,K}(S, s)$. Equation (A.15) makes this slope strictly increasing in s . Compactness and successor separation then give a strictly positive gap between the two sets of slopes. $\square$

Lemma 8 (Capital envelopes at the frontier). *Fix a classified primitive vector p and its frontier S . There are two local restricted Bellman values D^- and D^+ , represented on fixed compact sets of first actions, such that $V_p = \max\{D^-, D^+\}$ near S . Their active actions at S are precisely the first actions of the extinct and productive optimum sets $\mathcal{O}_-$ and $\mathcal{O}_+$, respectively. They satisfy*

$$D^\sigma(S + z) = V_p(S) + \max_{\pi \in \mathcal{O}_\sigma} a_\pi z + o(|z|), \quad \sigma \in \{-, +\}, \quad (\text{B.1})$$

where $a_\pi = \eta\{q + F_{p,K}(S, L_0)\}/C_0$. Consequently

$$V_p(S + z) = V_p(S) + \max_{\pi \in \mathcal{O}_- \cup \mathcal{O}_+} a_\pi z + o(|z|). \quad (\text{B.2})$$

The result allows multiple optimal histories and nondifferentiable optimal policies.

Proof. Write

$$\begin{aligned} c(K, L, s) &= qK + F_p(K, L) - s, \\ \mathcal{Q}_p(K, L, s) &= \eta \log c(K, L, s) + (1 - \eta) \log(1 - L) + \beta V_p(s). \end{aligned}$$

Initial capital affects first consumption; continuation value depends on s alone.

Fixed compact action sets. Lemma 7 and compactness give $d > 0$ such that every extinct active successor at S is at most $S - 2d$, and every productive one is at least $S + 2d$. Restrict the two Bellman problems to $s \leq S - d$ and $s \geq S + d$. Denote their values by $D^-(K)$ and $D^+(K)$. Both equal $V_p(S)$ at S .

Choose a common invariant resource cover $[0, \bar{K}]$ for initial stocks near S , as in Lemma 1. Give $\mathcal{Q}_p$ value $-\infty$ at zero consumption, zero successor or labor one. On the closed feasible first action sets this extension is upper semicontinuous: V_p is continuous at positive stocks, and $V_p(s) \leq E_p(s) + \mathcal{D}_p \rightarrow -\infty$ as $s \downarrow 0$. The restricted maxima are attained. An active action from each branch remains feasible when K is close to S and supplies a common finite lower bound on both restricted values.

There are therefore $c_0, s_0, \varepsilon > 0$ such that all nearby restricted maximizers have $c(K, L, s) \geq 4c_0$, $s \geq 2s_0$, and $L \leq 1 - \varepsilon$. Indeed, current reward and continuation value have common upper bounds on the resource cover. If consumption approached zero, the first logarithm would force the objective to $-\infty$; if the successor approached zero, the cake upper bound would do the same. As labor tends to one, the leisure logarithm also forces the objective to $-\infty$. Each possibility contradicts the finite lower bounds supplied by the feasible actions already identified.

Shrink the stock neighborhood so that $|c(K, L, s) - c(S, L, s)| \leq c_0$ uniformly for $0 \leq L \leq 1$. For each branch use the fixed compact index set

$$Q_\sigma = \{(L, s) : 0 \leq L \leq 1 - \varepsilon, s_0 \leq s \leq \bar{K}, \\ c(S, L, s) \geq 2c_0, s \text{ satisfies the branch cut}\}.$$

Every nearby restricted maximizer lies in its corresponding set, and every indexed action is feasible nearby with consumption at least c_0 . Hence $D^\sigma(K) = \max_{Q_\sigma} Q_p(K, L, s)$ exactly. Closedness of the unrestricted optimal successor graph excludes successors in $[S - d, S + d]$ for initial stocks sufficiently close to S . Bellman's equation then gives $V_p = \max\{D^-, D^+\}$. At S , each active action and optimal continuation form an optimal history in the corresponding classified branch.

Uniform expansion and active limits. On these compact sets, consumption is bounded away from zero. The CES production function has continuous first and second capital derivatives for $K > 0$, including its zero labor extension. Uniform Taylor expansion therefore gives

$$Q_p(S + z, L, s) = Q_p(S, L, s) + a(L, s)z + O(z^2), \quad a(L, s) = \frac{\eta\{q + F_{p,K}(S, L)\}}{c(S, L, s)},$$

with a common remainder bound. Only first consumption is differentiated; the finite, continuous continuation value is unchanged.

The active set on Q_σ is compact. As $z \rightarrow 0$, uniform convergence makes every limit of nearby maximizers active at S ; continuity of a makes their slopes approach the active slope set. Compare the upper bound from a nearby maximizer with the lower bound from an active maximizer of az . Convergence to the active slope set makes the difference between these bounds $o(|z|)$. This proves (B.1). Taking the maximum of the two expansions proves (B.2). $\square$

Proof of Theorem 2. Lemma 7 gives successor and slope separation. Lemma 8 gives (B.2) without assuming differentiability of value. The right derivative is therefore the largest active slope and the left derivative the smallest, proving (3.2).

Strong order puts successors chosen below S at or below every successor chosen at S , and those chosen above at or above every such successor. Closedness then gives the respective limits $\min G_p(S)$ and $\max G_p(S)$. The labor equation $s - qK = \zeta(L)$ has $\zeta'(L) = T_L > 0$ on its maximizing interior branch. Since the two limiting successors are distinct, continuity and $\partial L / \partial s = 1/T_L > 0$ give strictly ordered limiting labor.

If an optimal history reached S after its initial date, keep its preceding labor fixed and vary the successor slightly in both directions. Positive consumption makes both variations feasible. The successor objective is a smooth current payoff plus βV_p ; its right derivative exceeds its left derivative by $\beta J_p > 0$. It cannot have a local maximum there. Finally, the one sided expansions give (3.3) and, for each fixed $\tau \geq 0$, (3.4). The active slope formula and compactness give $0 < V'_{p,-}(S) < \infty$, proving the stated interval of admissible transfer losses. $\square$

Proof of Proposition 3. Fix a base vector p_0 and write $S_0 = S_{p_0}$. Take a compact convex primitive neighborhood inside the strict classification region, with a common invariant cover $[0, \bar{K}]$. Shrink it below as needed. All constants in this proof are uniform on this neighborhood. The continuity proof of Theorem 1 gives jointly continuous normalized values $\hat{V}_p(k) = V_p(\bar{K}k) - E_p(\bar{K}k)$, extended by $\hat{V}_p(0) = 0$, and $0 \leq \hat{V}_p \leq \bar{\mathcal{D}}$ for one finite $\bar{\mathcal{D}}$, chosen uniformly from the bounds $\mathcal{D}_p$.

Control in log capital. The compact control rectangle in that proof bounds optimal saving fractions by $z \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Writing $Q_p(K, L) = qK + F_p(K, L)$, every optimal first action therefore has $C \geq \varepsilon Q_p(K, L)$. CES factor shares imply

$$0 < \frac{KQ_{p,K}(K, L)}{Q_p(K, L)} \leq \theta_{\max}.$$

This holds also at $L = 0$, where $Q_p = qK$. Choose the stocks so that $e^{\theta_{\max}|\log(K_2/K_1)|} \leq 1 + \varepsilon/4$. If (L_i, s_i) is optimal at either endpoint K_i , then $s_i \leq (1 - \varepsilon)Q_p(K_i, L_i)$. For every intervening stock K , integration of the resource elasticity gives

$$\begin{aligned} \frac{Q_p(K_i, L_i)}{Q_p(K, L_i)} &\leq e^{\theta_{\max}|\log(K_2/K_1)|}, \\ \frac{Q_p(K, L_i) - s_i}{Q_p(K, L_i)} &\geq 1 - (1 - \varepsilon)(1 + \varepsilon/4) \geq \varepsilon/2. \end{aligned}$$

The action remains feasible throughout the interval. Holding it fixed changes only first consumption, whose payoff derivative satisfies

$$\left| \frac{d}{d \log K} u_p(Q_p(K, L_i) - s_i, L_i) \right| = \frac{\eta K Q_{p,K}(K, L_i)}{Q_p(K, L_i) - s_i} \leq \frac{2\eta_{\max}\theta_{\max}}{\varepsilon}.$$

Use the action optimal at K_1 as a feasible comparison at K_2 , and then reverse the comparison. The continuation value $V_p(s_i)$ is unchanged in each comparison, so Bellman's equation bounds the value difference in both directions. Partitioning any positive log stock interval into such neighborhoods gives

$$|V_p(K_2) - V_p(K_1)| \leq L_V |\log(K_2/K_1)|, \quad 0 < K_1, K_2 \leq \bar{K}.$$

Subtracting cake gives a uniform analogous bound $L_{\hat{V}}$ for $\hat{V}_p$.

Primitive sensitivity of normalized welfare. The functions $g_p(k, L)$ used in the continuity proof are uniformly Lipschitz in primitives, including on the axes. To check the only possible singularities, for $K, L > 0$ put $m = \min\{K, L\}$, $t = \max\{K, L\}/m$, and write the CES aggregate as

$$H_p(K, L) = m[a + (1 - a)t^\rho]^{1/\rho},$$

Here $a = w$ when $K \leq L$, and $a = 1 - w$ when $L \leq K$. The representations agree at $K = L$, where $t = 1$ and $H_p = m$. The bracket is bounded away from zero, and H_p/m is uniformly bounded. To check the weight and substitution derivatives, write

$$\begin{aligned} \partial_a \log(H_p/m) &= \frac{1 - t^\rho}{\rho\{a + (1 - a)t^\rho\}}, \\ \partial_\rho \log(H_p/m) &= -\frac{\log\{a + (1 - a)t^\rho\}}{\rho^2} + \frac{(1 - a)t^\rho \log t}{\rho\{a + (1 - a)t^\rho\}}. \end{aligned}$$

Both are uniformly bounded: in particular, $\sup_{t \geq 1} t^\rho \log t$ is uniformly finite since ρ stays strictly negative. The θ derivative satisfies

$$\frac{F_p(K, L)}{K} |\log H_p(K, L)| \leq C m^{\theta-1} (1 + |\log m|),$$

which is bounded because $m \leq 1$ and θ stays strictly above one. In fact, $m^{\theta_{\min}-1}(1 + |\log m|) \rightarrow 0$ as $m \downarrow 0$, where $\theta_{\min} > 1$ is the smallest degree on the primitive neighborhood. Thus these estimates control every primitive derivative of F_p/K uniformly near either axis; the zero labor and zero stock extensions preserve the resulting Lipschitz bound. Since $g_p \geq q_{\min} > 0$, it follows that $\|\log g_p - \log g_{\hat{p}}\|_{\infty} \leq C_g \|p - \hat{p}\|$.

The common compact control rectangle also contains maximizers of $\mathcal{T}_p \hat{V}_p$: all these continuation functions lie between zero and $\bar{D}$, so the same logarithmic penalties exclude boundary controls. On that rectangle the normalized current reward is uniformly Lipschitz in primitives. For $k > 0$, the two successor arguments have log ratio $\log(g_p/g_{\hat{p}})$; at $k = 0$ both are zero. The log capital bound and boundedness of $\hat{V}_p$ therefore give

$$\|\mathcal{T}_p \hat{V}_p - \mathcal{T}_{\hat{p}} \hat{V}_{\hat{p}}\|_{\infty} \leq L_T \|p - \hat{p}\|.$$

Contraction yields $\|\hat{V}_p - \hat{V}_{\hat{p}}\|_{\infty} \leq L_T \|p - \hat{p}\|/(1 - \beta_{\max})$. Adding back cake proves parameter Lipschitz continuity of V_p on every fixed positive stock interval. No uniform bound for the unnormalized values down to zero is asserted.

The branch crossing. Use the fixed successor cuts $S_0 - d$ and $S_0 + d$ from Lemma 8, and write $c_p(K, L, s) = Q_p(K, L) - s$. A base active action from each branch remains feasible nearby and gives a common finite lower bound on the restricted value. The uniform cake upper bound and logarithmic penalties therefore give $c_0, s_0, \varepsilon > 0$ such that every nearby restricted maximizer satisfies $c_p(K, L, s) \geq 4c_0$, $s \geq 2s_0$ and $L \leq 1 - \varepsilon$.

Shrink the neighborhood until $|c_p(K, L, s) - c_{p_0}(S_0, L, s)| \leq c_0$ uniformly on the common first action cover. Define the fixed compact sets by

$$Q_{\sigma} = \{(L, s) : 0 \leq L \leq 1 - \varepsilon, s_0 \leq s \leq \bar{K}, \\ c_{p_0}(S_0, L, s) \geq 2c_0, s \text{ satisfies the fixed branch cut}\}.$$

Every nearby restricted maximizer belongs to its set, since its base consumption is at least $3c_0$. Conversely, every indexed action remains feasible with consumption at least c_0 . The representation is therefore exact. Joint closedness of the unrestricted optimal graph excludes successors between the cuts, giving

$$D_p^{\sigma}(K) = \max_{(L,s) \in Q_{\sigma}} \{u_p(Q_p(K, L) - s, L) + \beta V_p(s)\}, \quad V_p(K) = \max\{D_p^-(K), D_p^+(K)\}.$$

The parameter bound just proved, and positive consumption on these sets, give $|D_p^{\sigma}(K) - D_{\hat{p}}^{\sigma}(K)| \leq L_D \|p - \hat{p}\|$.

At (p_0, S_0) every productive active capital slope exceeds every extinct one. Compactness, continuity and upper hemicontinuity of the two restricted maximizing sets preserve a uniform gap: nearby active slopes in the positive branch are at least a_+ and those in the negative branch at most a_- , with $m_0 = a_+ - a_- > 0$. The fixed compact representations make each branch locally Lipschitz in K , hence absolutely continuous on compact subintervals. At each differentiability point, the envelope argument places the branch's derivative within its active slope bounds. Integrating these almost everywhere bounds gives, for $\Delta_p = D_p^+ - D_p^-$,

$$\Delta_p(K_2) - \Delta_p(K_1) \geq m_0(K_2 - K_1), \quad K_2 > K_1$$

in a common frontier neighborhood. Frontier continuity keeps S_p there and strictly between the fixed successor cuts. No optimal successor lies between those cuts. At S_p , classification places the extinct successors below S_p and the productive successors above it; attained coexistence supplies both. Hence $D_p^-(S_p) = D_p^+(S_p) = V_p(S_p)$ and $\Delta_p(S_p) = 0$. Away from the frontier these branches remain first decision restrictions, not values conditional on an eventual regime. Comparing the roots for p and $\hat{p}$ now gives (3.5) with $C_S = 2L_D/m_0$. $\square$

C Barrier construction, completeness and margins

C.1 Calibrated finite block construction

Proof of Theorem 4. The monotone operator $\mathcal{F}_U$ preserves $[0, c_N]$. Since $C_N \leq c_N$, all iterates in (4.15) lie there. The cap can be read as an auxiliary stopping option: stop at cost c_N or continue along a staying edge. Unfolding $\mathcal{F}_U^N g_0$ gives the infimum over two classes: paths stopped after $n \in \{0, \dots, N-1\}$ edges, with cost $\sum_{t < n} \beta^t d_U(e_t) + \beta^n c_N$; and full N edge prefixes, with cost $\sum_{t < N} \beta^t d_U(e_t) + \beta^N C_N$. Nonnegative residuals make every early stop cost at least

$$\beta^n c_N = C_N \beta^{n+1-N} \geq C_N.$$

A full prefix costs at least $\Delta_{\text{blk}} + \beta^N C_N = C_N$. If continuation is impossible, the stopping option remains available. Infinite loss edges cannot reduce an infimum, and no attainment is required. Thus $g_N \geq g_0$. Apply monotonicity to each iterate separately, without interchanging $\mathcal{F}_U$ and a maximum, to obtain

$$\mathcal{F}_U \ell_U \geq \max_{1 \leq j \leq N} g_j \geq \max_{0 \leq j < N} g_j = \ell_U.$$

In particular $C_N \leq \ell_U \leq c_N$, and on every staying edge $\ell_U(x) \leq d_U(e) + \beta \ell_U(y)$. For an edge with finite reward, rearrange this inequality using the definition of d_U to obtain

$$B(x) = U(x) - \ell_U(x) \geq r(e) + \beta[U(y) - \ell_U(y)].$$

An edge with reward $-\infty$ satisfies the barrier inequality directly. Both U and ℓ_U are bounded, so the discounted terminal value of B tends to zero along every staying history. The witness calibration gives $W_M - B \geq \ell_U - \alpha \geq C_N - \alpha$. This staying margin and the independently established adverse comparisons give the full certificate. Theorem 3 then establishes selection.

For sharpness, fix any $0 \leq \alpha < C_N$ and put $\Delta = \Delta_{\text{blk}} - (1 - \beta^N)\alpha > 0$. Take N staging states $x_0, \dots, x_{N-1}$ on a cycle. The first $N-1$ staying edges earn zero; the last earns $-\Delta/\beta^{N-1}$. Each state also has free entry into a zero reward target absorber. Nonpositive rewards bound all welfare by zero, and the entry witnesses attain that bound. Thus $V = W_M = W_H = 0$. Use the regional reference $U \equiv \alpha$. A block starting at x_i has reference loss

$$\Delta \beta^{-i} + (1 - \beta^N)\alpha \geq \Delta_{\text{blk}}, \quad 0 \leq i < N.$$

The eternal cycle from x_0 has welfare

$$-\frac{\Delta}{1 - \beta^N} = -(C_N - \alpha).$$

Every valid regional barrier dominates this stayer by its terminal condition, while every feasible witness floor is at most zero. Consequently no uniform staying margin can exceed $C_N - \alpha$, and

the construction attains that bound. For $N > 1$ and $\alpha = 0$, this is an example with zero minimum single period residual and positive block loss. Finally add an edge from the target, earning $-\varepsilon_{\text{ad}}$, to an outside zero reward absorber. Its maximum adverse comparison slack is ε_{ad} : the target witness cannot exceed zero and the continuation ceiling cannot be below the outside value zero. The zero ceiling attains that slack. This proves sharpness of the full margin $\min\{C_N - \alpha, \varepsilon_{\text{ad}}\}$ for every admissible α . $\square$

C.2 Controlled renewal and shorter comparison tests

Proof of Proposition 5. Nonpositive rewards give the independent ceiling $\bar{V} = 0$, including after reopening. Immediate renewal followed by residence attains $W_M(i) = -I_i$; residence attains $W_H = 0$. On an ordinary edge $i \rightarrow j > i$,

$$B(i) + s - \beta B(j) = D_N[\beta^{N-j} - \beta^{N-1-i}] \geq 0.$$

On a maintenance edge,

$$B(i) + c_e - \beta B(j) = (c_e - s) - D_N[\beta^{N-1-i} - \beta^{N-j}] \geq c_e - c \geq 0.$$

The last bound uses $\beta^{N-1-i} - \beta^{N-j} \leq 1 - \beta^N$. The barrier is bounded, its terminal term vanishes, and $W_M(i) - B(i) \geq C_N - \bar{I}$. Departure slacks are $a - I_i \geq a - \bar{I}$ from M and d from h . Theorem 3 gives selection. An optimum exists by the finite discounted Bellman contraction: finite menus attain each maximum, and a maximizing stationary policy attains the fixed point by telescoping the bounded terminal value. This existence argument does not require computing that fixed point.

Every N staying edges contain maintenance. Ordinary costs contribute $s(1 - \beta^N)/(1 - \beta)$, and the maintenance premium contributes at least $\beta^{N-1}(c - s)$. This proves the block bound and calibration. To bound entry time, suppose an optimal history revisits a wear state $x_p = x_q = x$, with $p < q$. Its continuations are optimal: replacing a suboptimal tail would improve the history. All prefix rewards and values are finite. With $m = q - p$ and $R = \sum_{t=p}^{q-1} \beta^{t-p} r(e_t)$, the two optimal tails give $V(x) = R + \beta^m V(x)$. Repeating the observed staying cycle forever is feasible and earns $R/(1 - \beta^m) = V(x)$. This would be an optimal eternal stayer, contrary to selection. Thus every wear state is visited at most once, giving $\tau_H \leq N$. This argument allows ties and history dependent optimal choices; it needs neither a unique policy nor an acyclic feasible graph. The periodic optimum described below enters at date N from state zero, proving sharpness.

A constant barrier cannot establish the same conclusion. Such a barrier b_0 on an ordinary edge must satisfy $b_0 \geq -s + \beta b_0$, hence $b_0 \geq -T_s$. If a core history renews at action date $T \geq 0$, all its preceding rewards are at most $-s$, and its renewal cost is at least $I_{\min}$. Its welfare is therefore at most

$$-T_s(1 - \beta^T) - \beta^T I_{\min} = -T_s - \beta^T (I_{\min} - T_s) < -T_s.$$

Every feasible witness floor is below $-T_s$, so no constant barrier can satisfy weak domination. For a constant reference $U = u$, the smallest single period residual is $s + (1 - \beta)u$ because an ordinary edge exists. The best C_1 is $u + T_s$, whereas calibration to any feasible floor requires $\alpha \geq u - W_M(i) > u + T_s$. No admissible choice gives a positive calibrated margin.

For the exact solution of the periodic subcase, put $j = N - 1 - i$. The candidate in (4.19) satisfies $-s + \beta V(i + 1) = V(i) > -I$ for $i < N - 1$. At the last wear state, renewal exceeds maintenance by

$$\mathcal{L} = (c - s) - (1 - \beta^N)(I - T_s) > 0.$$

The strict inequality follows from $I < C_N < T_s + D_N$. Quitting earns $-a < V(i)$; residence and remaining outside dominate paid departure and reopening. Thus the candidate satisfies every Bellman comparison, strictly except for the designated optimal action. Bounded telescoping proves its value and the stated delay. For any $n < N$, a prefix from state zero can use only ordinary operation. With constant $U = u$, the minimum n period loss is $(1 - \beta^n)(u + T_s)$; calibration again exceeds $C_n = u + T_s$. With exact value as reference, all ordinary staying losses are zero, and the maintenance loss is $\mathcal{L}$. The minimum loss is zero for $n < N$ and $\beta^{N-1}\mathcal{L} > 0$ for $n = N$. The direct certificate does not depend on this exact value formula. $\square$

Proof of Corollary 2. The specified path costs $s(1 - \beta^{k_i})/(1 - \beta) + \beta^{k_i}I_{r_i} = Q_i$. It therefore supplies a complete feasible core witness. The staying inequalities and bounded terminal term proved above do not depend on renewal eligibility. Their witness gap is $W_M(i) - B(i) = D_N\beta^{N-1-i} - \beta^{k_i}(I_{r_i} - T_s) \geq \gamma$. Nonpositive rewards still give the unrestricted ceiling zero; adverse slacks are $a - Q_i$ and d . Selection follows from Theorem 3; finite menus and the no repeated state argument above give existence and the deadline. An ordinary edge still forces any constant barrier to be at least $-T_s$. Every core history renews at a finite action date T after paying at least s per period, so its welfare is at most $-T_s - \beta^T(I_r - T_s) < -T_s$. Weak witness domination is impossible.

For last state eligibility, substitution gives the displayed γ and $\max_i Q_i = I$. To verify the sharp boundary in the pure periodic cycle, start at the last state. One maintenance action followed by $N - 1$ ordinary operations costs $c + \beta s(1 - \beta^{N-1})/(1 - \beta)$ and returns there after N periods. Repeating the cycle costs $T_s + D_N$ in present value. At $I = T_s + D_N$, the bounded candidate $V(i) = -T_s - D_N\beta^{N-1-i}$ satisfies equality on all staying edges and on renewal at the last state. Quitting is strictly worse when $a > I$; staying at h and z dominates paid departure and reopening. Bellman telescoping verifies the candidate and the optimal eternal cycle, showing that not every optimum is core at the boundary. For $I > T_s + D_N$ and $a > I$, the same candidate still satisfies every Bellman comparison, with renewal now strictly worse than maintenance. The perpetual cycle remains optimal. For $I < T_s + D_N$, the strict certificate already proves selection. $\square$

C.3 Continuation accuracy and reward robustness

Proof of Proposition 6. If a history stays in M forever, its welfare is at most $B(x_0) \leq W_M(x_0) - \varepsilon$, contradicting (4.22). Otherwise a noncore history has a first adverse edge e_τ . Its continuation earns at most $r(e_\tau) + \beta\bar{V}(x_{\tau+1}) \leq W_i(x_\tau) - \varepsilon$, the same contradiction. Since $V \geq W_i$, the continuation regret condition implies the witness condition whenever the values are finite.

For every history with finite welfare, a bounded reward change shifts that welfare by an amount between $\underline{\psi}/(1 - \beta)$ and $\bar{\psi}/(1 - \beta)$; welfare $-\infty$ remains $-\infty$. Discounted positive rewards remain summable. Thus the shifted witnesses and unrestricted ceiling are valid. For each finite reward staying edge,

$$\tilde{B}(x) - \tilde{r}(e) - \beta\tilde{B}(y) = B(x) - r(e) - \beta B(y) + \bar{\psi} - \psi(e) \geq 0.$$

Infinite negative rewards satisfy the inequality directly. Bounded perturbations preserve exactly the histories with finite welfare, so the original terminal condition applies to every history required by the perturbed certificate. The added constant has a vanishing discounted terminal term. The staying comparison loses $\omega/(1 - \beta)$; the loss in an adverse comparison is at most

$$\psi(e) + \frac{\beta\bar{\psi} - \psi}{1 - \beta} \leq \frac{\omega}{1 - \beta}.$$

Selection follows from Theorem 3 when the residual margin is positive; at equality that theorem gives weak dominance.

For sharpness take $M = \{x\}$, $H = \{h\}$, with a staying reward $-(1 - \beta)\varepsilon$, free entry into h , and a zero reward target self loop. The inputs $B(x) = -\varepsilon$, $W_M = W_H = 0$ give margin ε and $V = 0$. Eternal staging has regret ε at every date, showing that equality in the accuracy threshold does not suffice. Increase the staging reward by $\bar{\psi}$ and change entry and target rewards by $\underline{\psi}$. Eternal staging and immediate entry earn

$$-\varepsilon + \frac{\bar{\psi}}{1 - \beta}, \quad \frac{\underline{\psi}}{1 - \beta}.$$

At $\omega = (1 - \beta)\varepsilon$, these and every finite entry date tie and are optimal. Above that bound eternal staging is strictly preferred. Hence neither universal strict threshold can be increased. $\square$

C.4 Compact completeness and sharp margins

Maintain the assumptions and exact loss profile of Subsection 4.7. Prefix classes contain every staying edge, including edges of infinite loss and prefixes that cannot continue indefinitely.

Lemma 9 (Compact loss profile). *One has*

$$\delta_T \uparrow G, \quad G = \sup_{T \geq 1} \frac{\delta_T}{1 - \beta^T}. \quad (\text{C.1})$$

If $G < \infty$, an eternal staying path attains it. An infinite zero loss staying path exists if and only if $\delta_T = 0$ for every finite T .

Proof. Matching successive endpoints is a closed condition. Each finite prefix class is therefore compact, and its lower semicontinuous loss attains a minimum whenever the class is nonempty. Nonnegative losses make δ_T nondecreasing. Put $g = \lim_T \delta_T$. Every eternal staying path has total loss at least g .

If $g < \infty$, the cylinders of compatible T edge prefixes with loss at most g are nonempty, closed and nested in $\mathcal{K}^{\mathbb{N}}$. Compactness gives a path in their intersection, whose total loss is at most g . Thus $G = g$, with attainment. If $g = +\infty$, every eternal staying path has infinite loss or none exists, so again $G = g$. Splitting an eternal path into blocks of T edges gives loss at least $\delta_T/(1 - \beta^T)$. This bound and $\delta_T \uparrow G$ prove the supremum formula, including empty path classes. Finally, the attained minimum is zero exactly when every finite prefix minimum is zero. $\square$

This compact extraction uses the finite horizon viability argument of Bertsekas (1972, Proposition 4). It converts the absence of an optimal eternal stayer into a positive finite block loss.

Proof of Theorem 5, part 1. Suppose every optimizer is core. A zero loss adverse edge has finite reward and finite successor value. Appending an attained optimal continuation would give a noncore optimum, a contradiction. Thus every adverse loss is positive. An eternal zero loss staying path would also attain value: telescoping its Bellman losses and using bounded $V|_M$ gives its welfare as V at its initial state. Lemma 9 therefore gives $G > 0$.

Fix a finite $0 < \gamma < G$. The compact loss profile supplies N and a finite $0 < \Delta_{\text{blk}} \leq \delta_N$ with $C_N = \Delta_{\text{blk}}/(1 - \beta^N) > \gamma$. Apply Theorem 4 with $U = V|_M$ and zero calibration error. Its bounded regional barrier has staying margin at least C_N . Selection and attainment provide core

witness floors $W_M = V|_M$ and $W_H = V|_H$. The exact ceiling $\bar{V} = V$ gives adverse slack $d(e) > 0$, completing a pointwise strict certificate.

If the adverse graph is nonempty and compact and its loss is lower semicontinuous, its positive losses have a positive minimum. An infinite minimum, or an empty adverse graph, permits any finite positive bound. Combining such a bound with C_N gives a uniform strict certificate. Theorem 3 proves both converses. $\square$

Proof of Theorem 5, part 2. Under selection, the preceding construction supplies a bounded barrier for every finite staying margin $0 < \gamma < G$. Margin zero uses $B = V|_M$, which satisfies the staying Bellman inequalities and has vanishing discounted terminal value.

If $G < \infty$, let an eternal staying path attain its loss G . Bounded $V|_M$ and telescoping give this path finite welfare $V(x) - G$ at its initial state x . Every valid regional barrier, bounded or not, dominates that welfare by its terminal condition. Consequently $V(x) - B(x) \leq G$, and no uniform staying margin with exact floors can exceed G . Together with the construction for every smaller finite margin, this proves $\Gamma_M = G$. When $G = +\infty$, the construction gives arbitrarily large finite staying margins and yields the same equality.

For a full certificate, every feasible witness floor satisfies $W_i(x) \leq V(x)$, and every legitimate continuation ceiling satisfies $\bar{V}(y) \geq V(y)$. Thus the staying comparison cannot exceed G at the minimizing path's initial state when $G < \infty$. On every adverse edge of finite loss,

$$W_i(x) - \{r(e) + \beta\bar{V}(y)\} \leq V(x) - \{r(e) + \beta V(y)\} = d(e).$$

Edges of infinite loss impose no finite upper bound. Taking the infimum gives $\Gamma \leq \min\{G, a_*\}$ without assuming that the adverse infimum is attained.

Conversely, exact core witnesses and exact adverse ceilings attain each adverse slack $d(e)$. A bounded barrier with staying margin greater than any prescribed finite $0 < \varepsilon < \min\{G, a_*\}$ therefore gives a full certificate of margin ε . There is always a weak certificate of margin zero using $B = V|_M$. These observations prove $\Gamma = \min\{G, a_*\}$, including $a_* = 0$, empty adverse graphs and infinite suprema. The formula specifies a supremum, without asserting endpoint attainment. $\square$

D Analytical comparisons

D.1 The patient harmonic economy

Proof of Theorem 6. Classification and feasible favorable returns. Here $D = \beta^{-1} - 1/2$ and $E = \beta^{-1} - 5/4 < 0$. The cover coefficient 8 exceeds $4\sqrt{2}$, so Theorem 1 applies for $A > A_*^{\text{gen}}(\beta)$, with $h_\beta < 1/2$.

Write $y_A(z) = A[2z/(1+z)]^{3/2}$, so $F_A(K, L) = L^{3/2}y_A(K/L)$. Its capital elasticity is $zy'_A(z)/y_A(z) = 3/[2(1+z)]$, decreasing from $3/2$ to zero. Since $1 < d_\beta < 3/2$, average product increases up to and beyond k_D , and $y_A(k_D)/k_D = Ac_D$. The bound $A > 1/(2c_D)$ gives $F_A(k_D, 1)/k_D > 1/2$. Thus the lower physical replacement root satisfies $\kappa_p < k_D$. For every $\kappa_p < K < k_D$,

$$F_A(K, 1) > K/2, \quad F_{A,K}(K, 1) > d_\beta/2 = D.$$

Both inequalities remain strict at a suitably chosen labor input below full capacity. The following construction makes that choice explicit.

Put $U = \min\{S_p, k_D\}$ and $K_0 = (\kappa_p + U)/2$. Classification gives $\kappa_p < S_p$, so $K_0 \in (\kappa_p, S_p)$ and both displayed inequalities hold at K_0 . Define

$$\begin{aligned} a_0 &= \left(\frac{K_0}{2A}\right)^{2/3}, & L_{\text{rep}} &= \frac{a_0 K_0}{2K_0 - a_0}, \\ t_E &= \left(\frac{D}{3\sqrt{2}A\sqrt{K_0}}\right)^{2/5}, & L_E &= \frac{K_0 t_E}{1 - t_E}, & L_{\dagger} &= \frac{1 + \max\{L_{\text{rep}}, L_E\}}{2}. \end{aligned} \quad (\text{D.1})$$

The full capacity inequalities imply $0 < L_{\text{rep}}, L_E < 1$, so all denominators are positive and $0 < L_{\dagger} < 1$. Indeed $F_A(K_0, L) > K_0/2$ exactly when $L > L_{\text{rep}}$, while

$$F_{A,K}(K_0, L) = 3\sqrt{2}A\sqrt{K_0} \left(\frac{L}{K_0 + L}\right)^{5/2} > D$$

exactly when $L > L_E$. Hence both strict screens hold at $(K_0, L_{\dagger})$. Continuity supplies a nonempty open interval I_p around K_0 , contained in (κ_p, S_p) , on which the same labor $L_{\dagger}$ satisfies both screens. Positive constant consumption $F_A(K, L_{\dagger}) - K/2$ is feasible there. Nevertheless every optimum becomes extinct by classification and therefore crosses below κ_p in finite time.

Global labor maximization. For the global comparisons, take $K > 0$ and $0 \leq s < K/2 + F_A(K, 1)$. Maximize the actual return over $0 \leq L \leq 1$, assigning $-\infty$ at $L = 1$:

$$r_A(K, s) = \max_{\substack{0 \leq L \leq 1 \\ K/2 + F_A(K, L) > s}} \left\{ \frac{1}{2} \log[K/2 + F_A(K, L) - s] + \frac{1}{2} \log(1 - L) \right\}.$$

At fixed $K > 0$, put $\zeta(L) = F_A(K, L) - (1 - L)F_{A,L}(K, L)$ and $j(t) = 1/2 - (5/2)t/(1 + t)$. Direct differentiation gives

$$\text{sign } \zeta'(L) = \text{sign} \left\{ \frac{2L}{1 - L} - j(L/K) \right\}.$$

The expression in braces increases strictly from $-1/2$ to infinity. Thus ζ first decreases and then increases. On the interval of positive consumption, the labor derivative of utility has the sign of $s - K/2 - \zeta(L)$. There is at most one positive interior local maximum, and it lies on the increasing branch of ζ . Utility tends to $-\infty$ at a positive lower consumption boundary and as $L \uparrow 1$. Only feasible zero labor can rival the interior maximum.

Define

$$\begin{aligned} K_* &= 4A, & L_* &= \frac{K_*}{3K_* + 2}, & F_* &= A \left(\frac{2K_*}{3(K_* + 1)} \right)^{3/2}, \\ C_0 &= \frac{4\sqrt{6}}{9} A \sqrt{\frac{K_*}{K_* + 1}}, & C_* &= \frac{F_*}{L_*}, & s_* &= \frac{K_*}{2} - C_0. \end{aligned} \quad (\text{D.2})$$

Then $C_0 = F_*(1 - L_*)/L_*$, $C_*(1 - L_*) = C_0$, and $(1 - L_*)F_{A,L}(K_*, L_*) = C_*$. Moreover,

$$\frac{2L_*}{1 - L_*} - j(L_*/K_*) = \frac{3K_* + 2}{6(K_* + 1)} > 0.$$

Thus L_* is the unique positive local maximum and ties zero labor exactly: the full maximizing set at (K_*, s_*) is $\{0, L_*\}$. Here $0 < L_* < 1/3$ and $8A/9 < s_* < 2A$. Every labor choice has positive consumption because even zero labor leaves cash $C_0 > 0$.

The saving kink and concavity violation. The positive maximum extends to a smooth branch near the contact because its labor second derivative is negative. Its utility advantage over zero labor has saving derivative $\frac{1}{2}(1/C_0 - 1/C_*) > 0$. The envelope therefore selects zero labor just below s_* and the positive branch just above it. The envelope's saving derivative jumps upward by $L_*^2/[2F_*(1 - L_*)] > 0$.

For a finite concavity comparison, put $d_* = F_*/2$. Evaluate zero labor at $s_* - d_*$ and L_* at $s_* + d_*$. These feasible choices bound the two endpoint maxima from below. At the contact, both branches attain the maximum. Hence

$$\frac{r_A(K_*, s_* - d_*) + r_A(K_*, s_* + d_*)}{2} - r_A(K_*, s_*) \geq \frac{1}{4} \log \left(1 + \frac{L_*^2}{4(1 - L_*)} \right) > 0. \quad (\text{D.3})$$

The product of the relative consumption factors is $(1 + d_*/C_0)(1 - d_*/C_*) = 1 + L_*^2/[4(1 - L_*)]$. Since $F_* < A$, one has $s_* - d_* > 7A/18$ and $s_* + d_* < 5A/2$. Zero labor is feasible even at the larger successor because $C_0/F_* = (1 - L_*)/L_* > 2$, hence $C_0 - d_* > 0$.

An increasing differences violation. Set $K_1 = K_*$, $K_2 = K_* + F_*$, $s_0 = s_*$ and $s_1 = s_* + F_*/4$. The zero labor cash balances are

$$\begin{aligned} K_1/2 - s_0 &= C_0, & K_1/2 - s_1 &= C_0 - F_*/4, \\ K_2/2 - s_0 &= C_0 + F_*/2, & K_2/2 - s_1 &= C_0 + F_*/4. \end{aligned}$$

Since $C_0/F_* > 2$, even the smallest is positive. Cash increases with K and decreases with s , so the full crossed rectangle is feasible at zero labor and therefore permits positive consumption at every labor choice. For every finite $K > 0$ and $0 < L < 1$,

$$\frac{F_A(K, L)(1 - L)}{L} < 2\sqrt{2}A\sqrt{L}(1 - L) \leq \frac{4\sqrt{6}}{9}A,$$

with the last maximum at $L = 1/3$. If zero labor leaves cash $c > 0$, it strictly dominates a given $L > 0$ exactly when $F_A(K, L)(1 - L)/L < c$. At the K_2 corner with the least remaining cash, that amount is

$$c_{\min} = C_0 + F_*/4 = \frac{4\sqrt{6}}{9}A\sqrt{x}(1 + x/8), \quad x = \frac{K_*}{K_* + 1}.$$

Because $A > 10$, $K_* > 8$, $x > 8/9$ and $x(1 + x/8)^2 > 800/729 > 1$. Both K_2 corners therefore select zero labor uniquely over the full labor range. At $L = 1$, welfare is $-\infty$. The contact gives the exact return at (K_1, s_0) , and the feasible L_* bounds the return at (K_1, s_1) from below:

$$\begin{aligned} & r_A(K_2, s_1) + r_A(K_1, s_0) - r_A(K_2, s_0) - r_A(K_1, s_1) \\ & \leq \frac{1}{2} \log \left[\frac{C_0 + F_*/4}{C_0 + F_*/2} \frac{C_*}{C_* - F_*/4} \right] = -\frac{1}{2} \log \left(1 + \frac{L_*^2}{8 - 6L_*} \right) < 0. \end{aligned} \quad (\text{D.4})$$

The preceding maximizations cover the full labor domain. The inequalities therefore compare actual envelope values. All comparison stocks exceed $7A/18 > h_\beta$ and are below $5A < 8A$.

Openness. Classification and frontier continuity persist on a neighborhood in the seven primitives. The physical root is simple because average product is increasing there, so it too is continuous. Fix a closed interval of positive length around K_0 inside I_p . Its strict physical and marginal return inequalities at the fixed $L_\dagger < 1$, and its location between κ_p and S_p , persist on a smaller primitive neighborhood.

At the finite comparison actions, positive zero labor cash gives a common consumption floor and a finite lower bound on each maximum. Resources bound consumption above. As $L \uparrow 1$, the leisure term tends to $-\infty$ uniformly on a small primitive neighborhood; all maximizing labor choices therefore lie in one fixed compact interval $[0, 1 - \varepsilon]$. Compact maximization makes these returns continuous, and both strict finite inequalities persist. The saving kink is established at the stated harmonic primitives. The two strict finite violations, rather than persistence of the kink, give the neighborhood conclusion. $\square$

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