

Minimum Viable Scale*

Bellman Barriers and the Economics of Survival

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Abstract

We study an economy with increasing returns to scale and endogenous labor. Capital that can be maintained may nevertheless be optimally liquidated. Under a patience condition and sufficiently high productivity, a unique minimum viable scale classifies every optimum: capital and consumption converge to zero below it and eventually remain bounded away from zero above it. Extinct and productive optima coexist at the frontier. The productive choices save more, work more and give capital a strictly higher marginal welfare value, creating an upward kink in continuous welfare. Along optimal extinction, labor vanishes more slowly than capital and output more quickly. Where every optimum survives, extinct histories can still approximate optimal welfare arbitrarily closely. We develop First Passage Bellman Barrier certificates for general discounted transition problems. They combine first passage analysis from physics with lower and upper bound methods from option pricing in finance. Strict comparisons establish finite target entry and permanence for every optimizer using feasible plans and independent welfare bounds, without solving the value function. A finite block construction lowers the barriers below feasible payoffs while preserving every regional Bellman inequality. Its guaranteed welfare margins are sharp for the stated bounds. Uniformly over an explicit parameter region, an analytical application certifies productive entry within 46 periods and permanence for every optimum from a specified initial capital range. Every constant regional Bellman barrier fails calibration against every feasible witness on that same range.

Keywords. minimum viable scale; optimal survival; increasing returns; endogenous labor; First Passage Bellman Barriers.

JEL codes. C61, D24, D61, O41.

1 Introduction

We study an economy with increasing returns to scale and endogenous labor. Under a patience condition and sufficiently high productivity, a unique minimum viable scale separates optimal extinction from productive survival. Below it, every optimum runs down capital, with capital and consumption converging to zero; above it, every optimum eventually sustains both bounded away from zero. Capital can be physically maintainable and still not be worth preserving: saving

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costs current consumption, and work costs leisure. At the frontier, extinct and productive optima coexist. Productive choices save more, work more and give additional capital a strictly higher marginal welfare value, creating an upward kink in continuous welfare.

We develop First Passage Bellman Barrier (FPBB) certificates for general discounted transition problems. The method combines first passage analysis from physics with lower and upper bound methods from option pricing in finance. Welfare arbitrarily close to optimal does not establish survival in the economy: postponing liquidation makes its discounted welfare cost arbitrarily small, even where every optimum survives. FPBB compares complete continuations at the first decision that violates the desired behavior. Feasible plans supply welfare lower bounds, while independent continuation ceilings cover every subsequent choice, including recovery. A regional Bellman barrier bounds the payoff from remaining indefinitely in a staging region, even as states and choices change. Our principal construction converts losses accumulated over finite blocks into a state dependent correction to the barrier. The correction preserves every staying Bellman inequality and lowers the barrier below a feasible plan's payoff at every staging state. Strict comparisons establish finite target entry and permanence for every optimizer without solving the value function.

Minimum viable scale exceeds the lower physical maintenance boundary. Between them, capital can sustain positive consumption and leisure forever yet is optimally liquidated. A small loss of usable capital near the frontier can therefore induce extinction while maintenance remains feasible. The classification does not assume steady state convergence. It persists under small changes in the model's parameters, and the frontier varies continuously with them.

Productivity determines the frontier's asymptotic scale. With the other primitives fixed in the patient region, minimum viable capital has order $A^{-1/(\theta-1)}$ as productivity $A \rightarrow \infty$, where θ measures returns to scale. Physical replacement gives a lower coefficient bound. A feasible productive expansion gives an upper coefficient bound governed by the return required to postpone consumption.

Production continues arbitrarily late into optimal extinction. Labor vanishes more slowly than capital, while output vanishes more quickly. Work becomes more intensive relative to the dwindling stock, yet production becomes too small to alter capital's leading depletion rate. These allocation laws require neither the patience condition nor the productivity restriction used for classification.

Critical capital stocks and optimal extinction are established subjects in nonconcave growth (Skiba, 1978; Majumdar and Mitra, 1982; Dechert and Nishimura, 1983). Akao, Kamihigashi, and Nishimura (2011) distinguish sustainable from critical capital, and Kamihigashi and Roy (2007) obtain global results without concave production. With endogenous labor, consumption and leisure preferences can generate multiple steady states and poverty traps even under convex technology (de Hek, 1998). We classify every optimum with endogenous work and leisure and derive allocation laws during extinction; the frontier comparison with an otherwise identical fixed labor economy remains open. Strict ordering of coexistence choices gives investment and labor jumps, and separation of welfare slopes gives the kink through established envelope and indifference principles (Milgrom and Segal, 2002; Wagener, 2020).

Bellman's principle and feasible continuation replacement underlie the verification argument (Bellman, 1957; Stokey and Lucas, 1989). First passage analysis in physics isolates the first boundary event (Redner, 2001). Storage inequalities bound accumulated flows by changes in a state function, as in an energy balance (Willems, 1972); the regional Bellman inequality applies this logic to welfare. In American option pricing, feasible exercise plans supply lower bounds, while dual upper bounds cover every exercise strategy (Haugh and Kogan, 2004). The calibrated construction gives welfare margins sharp for its stated bounds. Separate converses use exact value

under additional assumptions to characterize certificate existence and margin suprema. Section 6.5 relates these results to Bellman duality, finite step Lyapunov methods and discounted dissipativity.

An analytical application places the frontier within a factor of 1.798 of the lower maintenance boundary over an explicit productivity family. A complete FPBB certificate holds on a smaller parameter box within that family. Within that box, classification brings every optimum above the frontier to the certificate domain in finite time; the certificate and a finite prefix bound then give productive entry within 46 further periods and permanence thereafter. The initial delay has no explicit bound. Two period comparisons combine the current utility cost of saving more with the subsequent resource cost of saving less. These complementary losses produce a state dependent barrier that succeeds on a staging region where every constant regional Bellman barrier fails calibration against every feasible witness.

2 The economy and physical sustainability

Time is discrete and the horizon is infinite, $t = 0, 1, 2, \dots$. Population and the total labor endowment are normalized to one. Given current capital K , the planner chooses consumption C , labor $L \in [0, 1]$ and next period's capital K' . Leisure is $1 - L$. Output and undepreciated capital can both be consumed or carried into the next period. Minimum viable scale is the initial capital threshold for optimal preservation.

The resource constraint, technology, and period utility are

$$\begin{aligned} F_p(K, L) &= A\{wK^\rho + (1 - w)L^\rho\}^{\theta/\rho}, \\ C + K' &\leq qK + F_p(K, L), \quad C, K' \geq 0, \quad 0 \leq L \leq 1, \\ u_p(C, L) &= \eta \log C + (1 - \eta) \log(1 - L), \quad q = 1 - \delta. \end{aligned} \tag{2.1}$$

The primitive vector is $p = (A, \beta, q, \eta, \theta, \rho, w)$, where $A > 0$, $0 < \beta, q, \eta, w < 1$, $\rho < 0$, and $1 < \theta < 2$. Productivity A scales output; β discounts future utility; q is the retained fraction of capital; and η is the consumption weight in utility. The parameter w weights capital in the constant elasticity of substitution (CES) aggregate, ρ governs substitution with elasticity $1/(1 - \rho) < 1$, and θ measures returns when both inputs increase together. Primitive subscripts record dependence on the economy; calculations for a fixed economy may suppress them. Production is defined to be zero on either input axis, its continuous limit. We set $\log 0 = -\infty$. Preservation costs both consumption and leisure: saving reduces current consumption, and operating capital uses time. The marginal utility cost of work diverges as labor approaches one. Full labor is physically available but gives utility $-\infty$, so every finite welfare optimum leaves positive leisure and consumption. Zero labor remains admissible.¹ There is no additional labor cap or imposed upper bound on capital.

The restriction $\rho < 0$ makes capital and labor complementary, and both inputs are essential. Joint scaling gives $F_p(aK, aL) = a^\theta F_p(K, L)$ for $a > 0$: expanding both inputs raises output more than proportionally. The marginal product of capital need not rise with the stock. The fixed time endowment prevents indefinite joint expansion.

At fixed positive labor, as capital tends to zero, output is asymptotic to $Aw^{\theta/\rho}K^\theta$ and its capital derivative to $\theta Aw^{\theta/\rho}K^{\theta-1}$. Since $\theta > 1$, both output per unit of capital and its marginal product tend to zero. At the other extreme, increasing capital with labor bounded by one cannot raise output above $A(1 - w)^{\theta/\rho}$. Depreciation δK is proportional to capital at both extremes.

¹Without a leisure cost, raising labor at positive capital and consumption increases consumption at a fixed successor. Full labor would dominate every smaller input. Exogenously fixed labor instead removes this choice.

These properties produce two physical maintenance boundaries when productivity is high enough. The labor endowment creates the upper boundary; logarithmic leisure utility excludes full labor from finite welfare optima.

From initial capital K , a feasible history π specifies consumption, labor and next period's capital at every date, subject to the resource constraint. The planner maximizes $J_K(\pi) = \sum_{t \geq 0} \beta^t u_p(C_t, L_t)$. Write $V_p(K)$ for the value and $G_p(K)$ for the set of next period capital stocks chosen by optimal histories. A history is *extinct* if $K_t \rightarrow 0$ and *productive* if capital eventually remains in a bounded interval away from zero. This definition does not require convergence to a steady state; extinction need not mean immediate shutdown. Since $F_p(K, L) \leq A(1 - w)^{\theta/\rho}$, retention $q < 1$ bounds the resources available along every history from a fixed initial stock.

Define

$$\delta = 1 - q, \quad \nu = \frac{1 - \eta}{\eta}, \quad D = \beta^{-1} - q, \quad E = D - \theta\delta. \quad (2.2)$$

Saving sacrifices current consumption for retained capital and future output. For an interior candidate with constant consumption, the Euler condition requires marginal product D . Section 4.1 distinguishes this return requirement from the welfare comparison that determines minimum viable scale. The principal classification concerns the patient region

$$E < 0, \quad \text{equivalently} \quad \beta > [1 + (\theta - 1)(1 - q)]^{-1}. \quad (2.3)$$

Equivalently, $\beta[q + \theta\delta] > 1$. Our classification assumes this degree of patience relative to depreciation and increasing returns, together with sufficiently high productivity. The physical sustainability results below require neither restriction.

2.1 The complete sustainable set

The sustainable set contains all stocks that can be held constant with nonnegative consumption:

$$\mathcal{M}_p = \{K > 0 : F_p(K, 1) \geq \delta K\}.$$

The definition uses the largest physically available labor input. Holding capital constant with positive consumption and leisure requires a stock in the sustainable set's interior.

For $L > 0$, write $F_p(K, L) = AL^\theta y(K/L)$, where $y(k) = (wk^\rho + 1 - w)^{\theta/\rho}$. Here $k = K/L$ is the capital to labor ratio; at full labor it equals the capital stock. Put

$$k_a = \left[\frac{1 - w}{w(\theta - 1)} \right]^{1/\rho}, \quad H_a = \frac{y(k_a)}{k_a}, \quad A_{\text{rep}} = \delta/H_a. \quad (2.4)$$

The ratio k_a maximizes output per unit of capital. Maintenance is possible only if this maximum covers depreciation; it need not maximize lifetime welfare.

Proposition 1 (Physical sustainability). *The sustainable set is*

$$\mathcal{M}_p = \begin{cases} \emptyset, & A < A_{\text{rep}}, \\ \{k_a\}, & A = A_{\text{rep}}, \\ [\kappa_p, K_p^+], & A > A_{\text{rep}}, \end{cases} \quad (2.5)$$

where $\kappa_p < k_a < K_p^+$ are the two roots of $F_p(K, 1) = \delta K$. Positive constant consumption and leisure are feasible precisely at $K \in (\kappa_p, K_p^+)$ when $A > A_{\text{rep}}$. If $A < A_{\text{rep}}$, every feasible history obeys

$$K_t \leq (q + AH_a)^t K_0 \rightarrow 0. \quad (2.6)$$

The labor requirement for strict physical maintenance has lower boundary

$$\lambda_{\text{rep}} = \left(\frac{\delta}{AH_a} \right)^{1/(\theta-1)}.$$

Its productivity elasticity is $-1/(\theta - 1)$; the actual time endowment is fixed at one.

Proof. The share $s(k) = wk^\rho / (wk^\rho + 1 - w)$ strictly decreases from one to zero, and $ky'(k)/y(k) = \theta s(k)$. Thus $y(k)/k$ has its unique maximum H_a at k_a and vanishes at either end of the positive axis. This proves the three cases in (2.5), including the two roots above the threshold. Strict replacement at full labor remains strict at some labor below one by continuity. At a root, maintenance requires full labor and zero consumption. Outside the interval it is impossible. Homogeneity gives $F_p(K, L)/K \leq AH_a L^{\theta-1}$, with equality at $K = Lk_a$. This proves the contraction bound and shows that some stock permits strict maintenance at labor $L < 1$ exactly when $L > \lambda_{\text{rep}}$. Solving for the labor threshold and differentiating gives its stated elasticity. $\square$

Figure 1 shows the three maintenance cases.

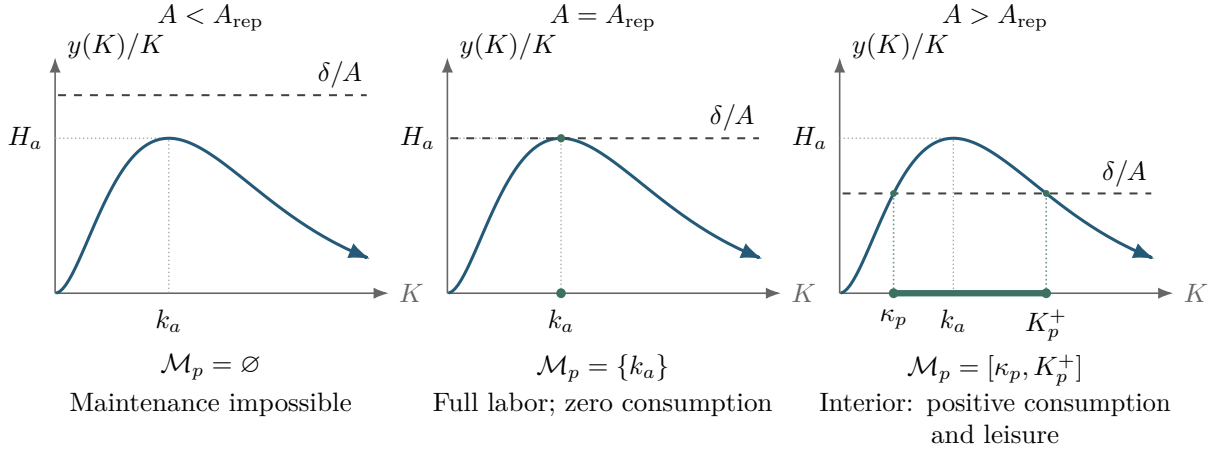


Figure 1: The complete sustainable set (Proposition 1). Holding other primitives fixed, higher productivity lowers δ/A against the same schematic curve. The singleton at tangency and both endpoints of the nondegenerate sustainable interval require full labor and zero consumption. Physical maintenance alone does not imply optimal survival.

The labor threshold allows the capital to labor ratio to vary; it is not a labor requirement at a fixed stock.

2.2 Physical maintenance and optimal extinction

At $A = A_{\text{rep}}$, maintaining the singleton stock requires zero consumption and full labor. Every optimum consumes a positive fraction of its capital and therefore runs the stock down.

Corollary 1 (Optimal extinction at the physical threshold). *If $A \leq A_{\text{rep}}$, there is an explicit constant $\gamma_p \in (0, 1)$ such that every optimum from every $K_0 > 0$ satisfies*

$$K_t \leq \gamma_p^t K_0 \longrightarrow 0. \quad (2.7)$$

Appendix A.2 gives γ_p and proves the consumption bound independently of productive capture or classification. At equality, the zero consumption maintenance path remains feasible but is not optimal. Above the threshold, physical sustainability leaves the welfare decision open.

3 Minimum viable scale and welfare

3.1 Minimum viable scale and optimal survival

Under (2.3), sufficiently high productivity gives a unique threshold separating optimal extinction from productive survival. The theorem classifies every optimum, including the choices at the threshold.

Theorem 1 (Minimum viable scale). *Fix the six primitives other than A and suppose (2.3) holds. The explicit construction in (A.12)–(A.13) gives a finite A_* and stocks $0 < b < h < \bar{K}(A)$. For every $A > A_*$, there is a unique frontier*

$$\kappa_p < S_p < b. \quad (3.1)$$

Every optimum from $0 < K < S_p$ becomes extinct. Every optimum above S_p enters $[h, \bar{K}(A)]$ in finite time and remains there. At S_p , attained extinct and productive optima coexist and exhaust the optimum set; their first successors lie strictly below and above S_p , respectively.

For $\kappa_p < K < S_p$, every optimal successor is below K ; for $S_p < K \leq b$, every optimal successor is above K . Every extinct optimum from $0 < K \leq S_p$ has a strictly decreasing capital path.

The classification persists under sufficiently small changes in all seven primitives within the patient region, and the frontier varies continuously with those primitives.

On (κ_p, S_p) in Figure 2, capital can be held constant with positive consumption and leisure, yet every optimum runs the stock down to zero. Preservation does not repay the consumption and leisure it costs. At the other end of the physical interval, K_p^+ limits constant maintenance, not survival: larger stocks can decline and survive.

On every productive optimal tail, consumption and leisure stay bounded away from zero, and output has a positive lower bound on every sufficiently long rolling window (Proposition 4). These guarantees allow multiple optima and need no steady state convergence.

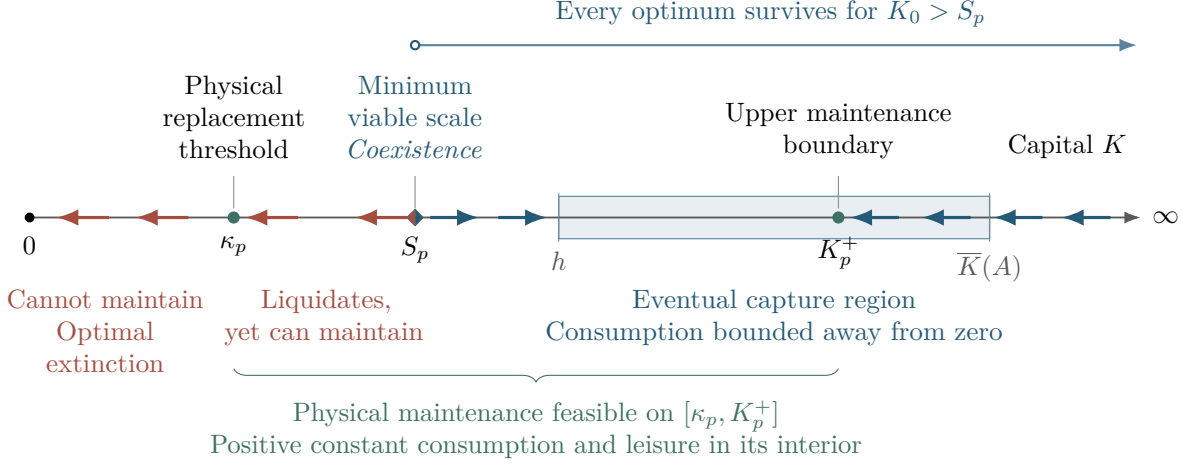


Figure 2: Physical maintenance and optimal survival (Theorem 1). Every optimum below S_p becomes extinct; extinct and productive optima coexist at S_p . Every optimum above S_p eventually remains in the shaded capture region. Arrows show the direction of next period's capital, allowing jumps; directions in $[h, K_p^+]$ are unspecified. Survival need not converge to a steady state. Distances are schematic (Proposition 1, Lemma 4).

The classification follows from a structural criterion (Lemma 3). Under the foundations of Lemma 1, suppose every optimum becomes extinct below one anchor, while every optimum above another enters in finite time and remains in a fixed compact region above both anchors. Strong successor order and exclusion of fixed optimal transitions between the anchors then give a unique coexistence frontier. Physical constraints force extinction at low stocks. A feasible reset and maintenance plan beats every low successor from high stocks, including unrestricted recovery. On the local interval, the labor proof separately excludes zero labor, identifies a unique interior maximizer conditional on the successor and establishes the mixed derivative sign that orders successors. The stationary equations exclude optimal transitions that hold capital fixed on that interval.

3.2 Choices and marginal welfare at the frontier

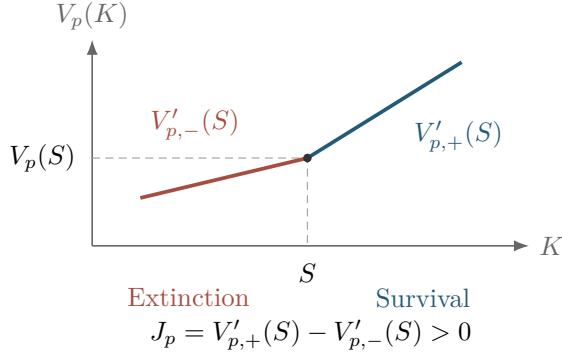
At the frontier, productive choices save more, work more and value additional capital more highly than extinct choices, although both attain the same welfare.

Theorem 2 (Frontier choices and welfare). *In an economy covered by Theorem 1 or Theorem 8, let $S = S_p$ and let $\mathcal{O}_-$ and $\mathcal{O}_+$ be its extinct and productive optimal history sets from S . These sets are compact. For each such history define $a_\pi = \eta\{q + F_{p,K}(S, L_0)\}/C_0$. Then*

$$\begin{aligned} \min_{\mathcal{O}_+} a_\pi &> \max_{\mathcal{O}_-} a_\pi, \\ V'_{p,-}(S) &= \min_{\mathcal{O}_-} a_\pi, \quad V'_{p,+}(S) = \max_{\mathcal{O}_+} a_\pi. \end{aligned} \tag{3.2}$$

In particular, $J_p := V'_{p,+}(S) - V'_{p,-}(S) > 0$. Every optimal successor selection jumps upward at S , with limits $\min G_p(S) < S < \max G_p(S)$. The corresponding labor limits are strictly ordered. No optimal history reaches S after its initial date.

At a fixed current stock, a larger successor induces more labor on the maximizing branch. The proof separately establishes strict increasing differences after labor is maximized: a larger successor makes additional current capital more valuable at the margin. This property orders the marginal value of capital across the two regimes. In a_π , the factor $q + F_{p,K}(S, L_0)$ measures the extra current resources supplied by an additional unit of initial capital at the history's chosen labor. Multiplying by η/C_0 , the marginal utility of consumption, gives its welfare value. The strict ordering in (3.2) produces the upward kink in Figure 3, even when either regime contains multiple optima.



A small transfer selects opposite regimes

The recipient moves to $S + u$: every optimum survives.

The donor moves to $S - u$: every optimum becomes extinct.

Their combined welfare rises.

Figure 3: Continuous welfare has a strict upward kink at $S = S_p$, even with multiple optima (Theorem 2). Lines show local tangent directions. The welfare gain concerns a sufficiently small costless transfer between identical economies initially at S ; see (3.3).

Consider two independent economies with identical primitives, both initially at S . A sufficiently small costless transfer of initial capital $u > 0$ from one to the other gives

$$V_p(S + u) + V_p(S - u) - 2V_p(S) = J_p u + o(u) > 0. \quad (3.3)$$

A Pareto improvement requires compensation.

4 Threshold determinants and policy burdens

4.1 Replacement and required returns

The physical and welfare frontiers impose different requirements on capital. At a maintenance boundary, output at full labor exactly replaces depreciation: $F_p(K, 1) = \delta K$. An optimal investment must also justify postponing consumption. For an interior candidate with constant consumption, the Euler condition requires the marginal product

$$D = \delta + \frac{1 - \beta}{\beta}. \quad (4.1)$$

The second term is the return required for postponement. At the lower maintenance boundary, output per unit of capital equals δ . As productivity rises and that boundary approaches zero, the marginal product at full labor approaches $\theta\delta$. The patient condition $E < 0$ therefore puts this limiting marginal product above the required return D . Neither comparison locates S_p : minimum viable scale compares complete histories, including the leisure forgone to operate capital.

Holding technology shape fixed, implicit differentiation of $Ay(K^*) = \delta K^*$ at either simple maintenance boundary gives

$$\frac{\partial \log K^*}{\partial \log A} = -\frac{1}{\theta s(K^*) - 1}, \quad \frac{\partial \log K^*}{\partial \log \delta} = \frac{1}{\theta s(K^*) - 1}, \quad s(K) = \frac{wK^\rho}{wK^\rho + 1 - w}. \quad (4.2)$$

Indeed, $d \log A + [\theta s(K^*) - 1] d \log K^* = d \log \delta$. The denominator is positive at κ_p and negative at K_p^+ . Higher productivity therefore expands the maintainable interval; higher depreciation contracts it. In Figure 1, these changes lower or raise the replacement line δ/A . In the depreciation comparison, retention changes as $q = 1 - \delta$. At tangency, $\theta s = 1$ and both boundary sensitivities diverge in absolute value.

Patience affects the investment requirement without changing physical maintenance. Consumption and leisure preferences determine the welfare cost of saving and working; they do not enter the maintenance equation. Complementarity and factor weights shape the input bottlenecks, while increasing returns determine how much additional scale raises output relative to capital. The physical effects of productivity and depreciation do not by themselves order the welfare frontier.

4.2 Productivity and minimum viable scale

Higher productivity changes the scale at which preservation becomes worthwhile. The lower maintenance boundary supplies a lower bound on minimum viable capital. For an upper bound, a feasible expansion must beat every history that remains below a suitable small stock. The two bounds have the same productivity exponent.

Theorem 3 (Productivity scaling). *Fix the six primitives other than A and suppose the patient condition (2.3) holds. Put $r = \theta - 1$ and $c_K = w^{\theta/\rho}$. There are finite constants $A_M \geq A_*$ and $c_+ > 0$ such that, for every $A > A_M$,*

$$\left(\frac{\delta}{c_K}\right)^{1/r} A^{-1/r} < \kappa_A < S_A < c_+ A^{-1/r}. \quad (4.3)$$

Consequently

$$S_A = \Theta\left(A^{-1/(\theta-1)}\right), \quad (4.4)$$

and the lower physical boundary satisfies

$$A^{1/r} \kappa_A \longrightarrow (\delta/c_K)^{1/r}. \quad (4.5)$$

More precisely,

$$\left(\frac{\delta}{c_K}\right)^{1/r} \leq \liminf_{A \rightarrow \infty} A^{1/r} S_A \leq \limsup_{A \rightarrow \infty} A^{1/r} S_A \leq \left(\frac{D}{c_K}\right)^{1/r}, \quad (4.6)$$

and

$$\limsup_{A \rightarrow \infty} \frac{S_A}{\kappa_A} \leq \left(\frac{D}{\delta}\right)^{1/r} = \left(1 + \frac{1 - \beta}{\beta \delta}\right)^{1/r}. \quad (4.7)$$

Here Θ means that two positive constants, independent of A , bound $A^{1/(\theta-1)} S_A$ above and below for all sufficiently large A . At fixed positive labor, output per unit of capital near zero is asymptotic to $A c_K K^{\theta-1}$. Higher productivity can support a smaller stock: keeping this average product of constant order requires capital of order $A^{-1/(\theta-1)}$. The theorem establishes that optimal survival has this scale when labor is chosen endogenously.

Replacement determines the lower coefficient bound. For the upper bound, Appendix D constructs a feasible stationary allocation whose average product approaches D . A feasible expansion beats an affine Bellman bound on welfare from remaining below its stock, after paying the consumption and leisure costs. Since an extinct optimum from that stock would decrease strictly, the comparison places it above the frontier. The benchmark is not assumed to be a stationary optimum; its average product approaching D does not impose the stationary Euler condition on the frontier.

Greater patience lowers the upper coefficient bound and tightens the bound on S_A/κ_A . Higher depreciation raises both coefficient bounds while lowering this relative bound; a smaller bound on relative separation does not imply a lower minimum viable stock. The theorem leaves monotonicity at finite productivity and the limiting coefficient of S_A open. Consumption and leisure preferences can still affect its position within these bounds.

4.3 Policy burdens and productive survival

We consider an initial loss of usable capital and a permanent loss of productive efficiency. Compliance uses labor (Trebbi, Zhang, and Simkovic, 2026); distortions in input allocation and regulation tied to producer size can lower productivity (Restuccia and Rogerson, 2008; Garicano, Lelarge, and Van Reenen, 2016). These experiments measure net resource costs. Applying them to taxes or regulation requires accounting for returned revenue and productive benefits. A fall in an asset's market price alone does not reduce capital in this model.

Initial capital burdens. Suppose a prescribed fraction χ of the available capital is lost before continuation choices. The remaining stock is $(1 - \chi)K_0$, with $0 \leq \chi < 1$; future primitives are unchanged.

Corollary 2 (Capital burdens and optimal survival). *Let p satisfy Theorem 1 or Theorem 8. The minimum initial stock measured before the loss is*

$$\mathcal{S}_p(\chi) = \frac{S_p}{1 - \chi}. \quad (4.8)$$

Every optimum is productive for $K_0 > \mathcal{S}_p(\chi)$ and extinct for $0 < K_0 < \mathcal{S}_p(\chi)$; both regimes attain welfare at equality. For $K_0 > \mathcal{S}_p$, the critical burden is

$$\chi^*(K_0) = 1 - \frac{S_p}{K_0}. \quad (4.9)$$

If

$$\chi^*(K_0) < \chi < 1 - \frac{\kappa_p}{K_0}, \quad (4.10)$$

every optimum becomes extinct although constant capital with positive consumption and leisure remains feasible after the loss.

Proof. Apply classification to $(1 - \chi)K_0$. The critical burden leaves exactly S_p . Condition (4.10) places the remaining stock strictly between κ_p and S_p , inside the sustainable interval. This burden interval is nonempty because $\kappa_p < S_p$. $\square$

The frontier in usable capital is unchanged. Since $\chi^*(K_0) \rightarrow 0$ as $K_0 \downarrow S_p$, arbitrarily small losses can induce extinction while maintenance remains feasible (Figure 2). The productive tail in Proposition 4 is then lost: $K_t \rightarrow 0$ and $F_p(K_t, L_t) \leq Aw^{\theta/\rho} K_t^\theta \rightarrow 0$.

For $K_0 > S_p$, write $W(\chi; K_0) = V_p((1 - \chi)K_0)$. Theorem 2 gives

$$W'_-(\chi^*; K_0) = -K_0 V'_{p,+}(S_p) < -K_0 V'_{p,-}(S_p) = W'_+(\chi^*; K_0) < 0. \quad (4.11)$$

Raising χ lowers usable capital, reversing the derivative sides: W'_- uses $V'_{p,+}$ and W'_+ uses $V'_{p,-}$. The marginal welfare loss is larger on the productive side; value remains continuous at the switch.

Permanent efficiency losses. Now let $A_e = (1 - \tau)A$, holding the other primitives fixed. This changes the technology at every date. The next result requires neither the patience condition nor the productivity restriction used for classification.

Proposition 2 (Physical vulnerability to efficiency losses). *Suppose $A > A_{\text{rep}}$ and $0 < \tau < 1$. If $A_e > A_{\text{rep}}$, then*

$$\kappa(A) < \kappa(A_e) < k_a < K^+(A_e) < K^+(A). \quad (4.12)$$

Every stock in $(\kappa(A), \kappa(A_e)]$ previously permitted constant capital with positive consumption and leisure. After the loss, every finite welfare history from that band becomes extinct. If

$$\tau > \tau^* := 1 - \frac{A_{\text{rep}}}{A}, \quad (4.13)$$

every feasible history satisfies $K_t \leq (q + A_e H_a)^t K_0 \rightarrow 0$. At $\tau = \tau^*$, every optimum becomes extinct, while full-labor, zero-consumption maintenance remains feasible at k_a .

The higher replacement line δ/A_e contracts the maintainable interval (Figure 1). At its new lower boundary, positive consumption forces the successor below the interval, explaining the boundary's inclusion. Loss of the upper band need not cause extinction: capital can decline into the remaining interval. Appendix A.3 proves the result.

The labor requirement in Proposition 1 satisfies

$$\lambda_{\text{rep}}(A_e) = \lambda_{\text{rep}}(A)(1 - \tau)^{-1/(\theta-1)}. \quad (4.14)$$

For fixed τ , the multiplication factor rises as θ falls toward one. The capital to labor ratio can adjust, but the time endowment remains one.

These results establish physical threshold shifts and a sufficient efficiency loss for global extinction. They do not determine whether smaller productivity changes raise or lower S_A .

5 Extinction dynamics

The allocation near zero explains how production can continue as capital vanishes. The results hold throughout (2.1), without the global classification's patience and productivity restrictions.

With zero labor, the policy $K_{t+1} = \beta q K_t$ and $C_t = (1 - \beta)q K_t$ consumes the stock gradually. We call it the cake policy. Its welfare is

$$E_p(K) = \frac{\eta}{1 - \beta} \log[(1 - \beta)qK] + \frac{\eta\beta}{(1 - \beta)^2} \log(\beta q). \quad (5.1)$$

Work raises welfare above cake, but as production becomes small relative to retained capital, optimal saving and consumption approach their cake proportions.

Put

$$\begin{aligned} g_c &= \beta q, & m_K &= Aw^{\theta/\rho}, & \gamma_E &= \frac{\theta - 1 - \rho}{1 - \rho}, \\ a_E &= \frac{\eta m_K}{(1 - \beta)q[1 - \beta g_c^{\theta-1}]}, & \ell_E &= \left[\frac{\theta m_K(1 - w)}{wq\nu(1 - \beta)} \right]^{1/(1-\rho)}. \end{aligned} \quad (5.2)$$

Here $\nu = (1 - \eta)/\eta$ as before. The constant m_K is the coefficient of output near zero capital when labor is large relative to capital, and $\theta - 1 < \gamma_E < 1$.

Theorem 4 (The path to extinction). *At every primitive vector in (2.1), as $K \downarrow 0$,*

$$\begin{aligned} V_p(K) - E_p(K) &\sim a_E K^{\theta-1}, & L &\sim \ell_E K^{\gamma_E}, & F_p(K, L) &\sim m_K K^\theta, \\ \frac{K'}{K} &\longrightarrow g_c, & \frac{C}{K} &\longrightarrow (1 - \beta)q. \end{aligned} \quad (5.3)$$

The action limits hold uniformly over all optimal actions. In particular, every optimum uses positive labor and produces at every sufficiently small positive stock, with $L \rightarrow 0$ and $L/K \rightarrow \infty$.

For every extinct optimal history π from a positive initial stock, there is a constant $\Xi_\pi \in (0, \infty)$ such that

$$\begin{aligned} \frac{K_t}{g_c^t} &\longrightarrow \Xi_\pi, & \frac{C_t}{g_c^t} &\longrightarrow (1 - \beta)q\Xi_\pi, \\ \frac{L_t}{g_c^{\gamma_E t}} &\longrightarrow \ell_E \Xi_\pi^{\gamma_E}, & \frac{F_p(K_t, L_t)}{g_c^{\theta t}} &\longrightarrow m_K \Xi_\pi^\theta. \end{aligned} \quad (5.4)$$

The proof is in Appendix C. Labor vanishes more slowly than capital, so labor intensity rises and capital becomes the binding input. This gives $F_p(K_t, L_t)/K_t^\theta \rightarrow m_K$, the same leading coefficient as at fixed positive labor, even though hours vanish. But $F_p(K_t, L_t)/K_t \rightarrow 0$ because $\theta > 1$: production becomes too small relative to the stock to change the leading depletion rate βq . Figure 4 shows this ordering. Patience and retention determine the leading rate of capital and consumption; productivity and complementarity affect the coefficients and path level, and complementarity also affects the labor exponent γ_E .

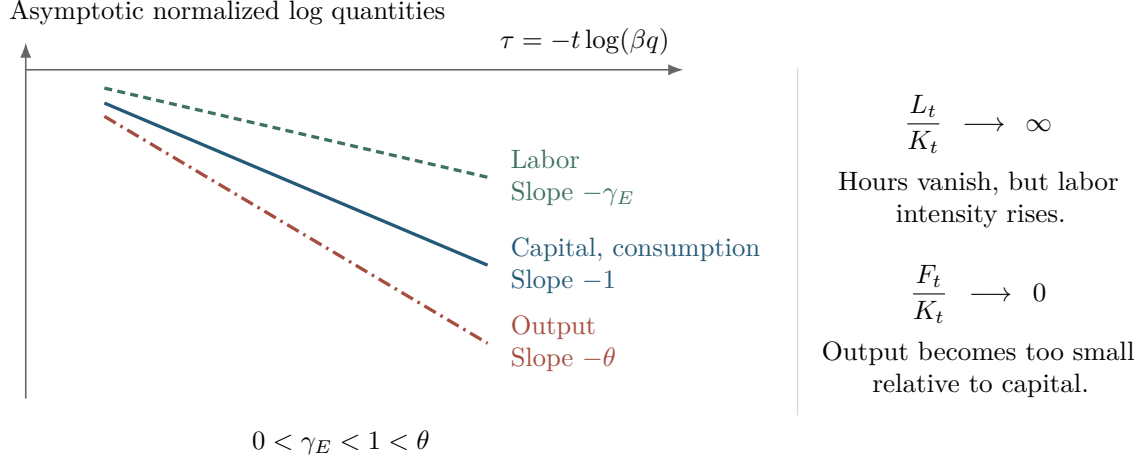


Figure 4: Extinction with continuing production (Theorem 4). Divide each quantity by its positive leading coefficient in (5.4). Against $\tau = -t \log(\beta q)$, the normalized logarithms have asymptotic slopes $-\gamma_E$ for labor, -1 for capital and consumption, and $-\theta$ for output. The rays illustrate this ordering, not finite date paths. The result holds throughout the primitive family, without the classification's patience and productivity restrictions.

6 First Passage Bellman Barriers

The verification problem is whether every optimum eventually enters a specified region and stays there. FPBB addresses this question on general discounted transition graphs. In the growth model, initial welfare accuracy cannot identify survival. A first passage comparison and a separate bound on perpetual waiting establish the behavior; finite block losses supply a calibrated barrier.

6.1 Why initial welfare accuracy does not identify survival

The cake policy in (5.1) has zero labor and $K_{t+1} = \beta q K_t$, so capital converges to zero. The resource bound $R_p = q + A H_a$ limits its welfare shortfall uniformly over positive stocks.

Proposition 3 (Uniform approximation by extinction). *At every primitive vector in (2.1), optimal histories exist from every $K > 0$, and*

$$0 \leq V_p(K) - E_p(K) \leq \mathcal{D}_p = \frac{\eta}{(1 - \beta)^2} \log(R_p/q). \quad (6.1)$$

Following any optimum for n dates and then switching permanently to cake gives an extinct history $\pi^{[n]}$ with

$$0 \leq V_p(K) - J_K(\pi^{[n]}) = \beta^n \{V_p(K_n) - E_p(K_n)\} \leq \beta^n \mathcal{D}_p. \quad (6.2)$$

Thus the supremum of extinct welfare equals $V_p(K)$ at every positive stock, including stocks from which every optimum survives.

Late switching makes the loss uniformly small. Where every optimum survives, the extinct supremum is unattained. The construction asserts neither a stationary policy nor a uniformly small Bellman residual. Appendix A gives the proof.

An optimal history must use an optimal continuation when the departure date arrives. A feasible alternative that earns more from that state rules out the departure, however small its date zero cost.

6.2 A first passage certificate

A feasible plan supplies a welfare floor. A regional Bellman barrier bounds welfare from remaining forever in the staging region, and an independent ceiling bounds continuation after an adverse departure. That continuation may recover and enter the target, so the ceiling must cover more than paths that remain outside. Strict comparisons with the floor exclude perpetual staging and adverse departure.

Let $\mathcal{G} \subseteq \mathbf{X} \times \mathbf{A} \times \mathbf{X}$ be the complete feasible transition graph. An edge $e = (x, a, y)$ earns $r(e) \in \mathbb{R} \cup \{-\infty\}$, discounted by $\beta \in (0, 1)$. Write $J_x(\pi)$ for the welfare of a feasible infinite history and $V(x) = \sup_{\pi} J_x(\pi)$. Assume discounted positive rewards are summable along every history, so welfare and finite prefix decompositions are well defined. No topology or existence of an optimizer is required.

Fix disjoint staging and target sets M, H . A history is *core* if it stays in M until finite entry into H and then remains in H ; entry occurs at date zero from H . Let $\mathcal{C}(x)$ denote these histories. The adverse edges are

$$\begin{aligned} E_M &= \{(x, a, y) \in \mathcal{G} : x \in M, y \notin M \cup H\}, \\ E_H &= \{(x, a, y) \in \mathcal{G} : x \in H, y \notin H\}. \end{aligned} \tag{6.3}$$

Perpetual waiting takes neither kind of edge and requires its own bound.

Definition 1 (First passage certificate). A weak First Passage Bellman Barrier consists of a real function B on M , an independently established ceiling $\bar{V} \geq V$ at every adverse successor, and finite welfare floors W_M, W_H such that:

1. Every staying edge satisfies

$$B(x) \geq r(e) + \beta B(y), \quad x, y \in M, \tag{6.4}$$

and every finite welfare history staying in M satisfies $\liminf_{T \rightarrow \infty} \beta^T B(x_T) \geq 0$.

2. At each $x \in M$, a specified core history earns at least $W_M(x)$; at each $x \in H$, a specified history remaining in H earns at least $W_H(x)$.
3. On the complete feasible graph, including boundary actions,

$$\begin{aligned} W_M(x) &\geq B(x) & (x \in M), \\ W_M(x) &\geq r(e) + \beta \bar{V}(y) & (e \in E_M), \\ W_H(x) &\geq r(e) + \beta \bar{V}(y) & (e \in E_H). \end{aligned} \tag{6.5}$$

All extended real expressions must be defined. The certificate is *strict* when the three comparisons are pointwise strict, and has margin $\varepsilon \geq 0$ when every comparison slack is at least ε .

Theorem 5 (Global selection). *A weak certificate makes every history from $M \cup H$ weakly dominated by a core history. Hence*

$$V(x) = \sup_{\pi \in \mathcal{C}(x)} J_x(\pi) = \sup_{\pi \text{ eventually remains in } H} J_x(\pi), \quad x \in M \cup H. \quad (6.6)$$

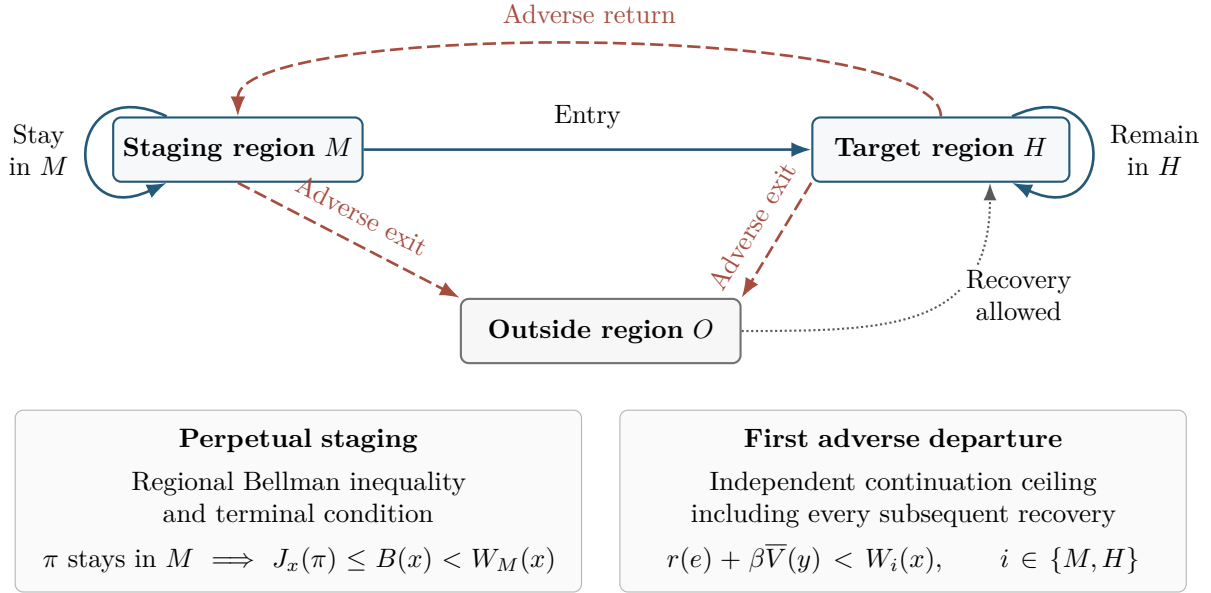
An unrestricted optimum, when it exists, has a core optimal replacement. A strict certificate makes every optimizer core.

Proof. A history of welfare $-\infty$ is dominated by its finite welfare witness. For a finite welfare history staying in M , iteration gives

$$\sum_{t < T} \beta^t r(e_t) \leq B(x_0) - \beta^T B(x_T).$$

The terminal condition therefore bounds its welfare by $B(x_0) \leq W_M(x_0)$. Every other noncore history has a first adverse edge at a finite date τ . Its continuation from that date is at most $r(e_\tau) + \beta \bar{V}(x_{\tau+1})$. Replace the edge and its tail by the witness at x_τ . The unchanged prefix and the witness form a core history with weakly greater welfare. Strict comparisons give strict improvement. Core histories already dominate themselves, and their inclusion among histories eventually remaining in H gives the value equalities. Replacing an attained optimum preserves attainment. $\square$

Figure 5 separates the transition classes and the two comparisons that exclude noncore histories.



Every optimizer enters H in finite time and remains there.

Figure 5: Strict FPBB selection from $M \cup H$ (Theorem 5). Here $O = \mathbf{X} \setminus (M \cup H)$. The feasible witness supplies W_i ; the two comparisons exclude perpetual staging and a first adverse departure. Arrows denote transition classes when feasible. Dashed adverse departures are inferior under the certificate. The ceiling covers every subsequent recovery, including the dotted route. A loop in H denotes movement within the target, not a stationary state.

Remark (Continuation accuracy). If a certificate has margin $\varepsilon > 0$, a finite welfare history from $M \cup H$ is core whenever its continuation at every visited regional state $x_t \in i$, $i \in \{M, H\}$, satisfies $J_{x_t}(\pi_t) \geq W_i(x_t) - \xi$ for one $0 \leq \xi < \varepsilon$. Indeed, eternal staging earns at most $B(x_0) \leq W_M(x_0) - \varepsilon$; a first adverse edge earns at most $W_i(x_t) - \varepsilon$, including its continuation. Either possibility contradicts the stated accuracy. When value is finite, continuation regret at most ξ suffices since $V \geq W_i$. This condition concerns every visited state, not merely initial welfare; it supplies no particular entry deadline.

6.3 Calibrated construction from finite block losses

A regional reference may bound perpetual staying yet exceed every available feasible payoff. Calibration requires lowering it by more than a witness's shortfall. Summing finite block losses bounds staying welfare, but a usable correction must also preserve every one period Bellman inequality, including at states with zero immediate loss.

Let $M \neq \emptyset$. Take a bounded regional reference $U : M \rightarrow \mathbb{R}$. Define its residual as the current reference less current reward and the discounted successor reference. Require a nonnegative residual on every staying edge:

$$d_U(e) = U(x) - r(e) - \beta U(y) \geq 0, \quad x, y \in M. \quad (6.7)$$

Allow $d_U(e) = +\infty$ when $r(e) = -\infty$. A valid certificate satisfies

$$B \leq W_M \leq V \quad \text{on } M, \quad W_H \leq V \quad \text{on } H,$$

while $V \leq \bar{V}$ at adverse successors. The reference need not bound unrestricted value or extend beyond M .

Fix an integer $N \geq 1$ and a finite $\Delta_{\text{blk}} > 0$. Suppose every N edge staying prefix satisfies

$$\sum_{t < N} \beta^t d_U(e_t) \geq \Delta_{\text{blk}}. \quad (6.8)$$

The test includes prefixes that cannot be extended indefinitely; an empty prefix class satisfies it.

The quantity C_N below bounds total discounted reference loss from below along any perpetual staying path, by summing the block bounds. An auxiliary stopping cost keeps the calculation bounded even when no staying continuation exists. The stopping choice belongs only to the calculation: it neither changes the feasible graph nor represents target entry. For a bounded real function z on M and $x \in M$, define

$$C_N = \frac{\Delta_{\text{blk}}}{1 - \beta^N}, \quad c_N = C_N \beta^{1-N}, \quad (6.9)$$

$$(\mathcal{F}_U z)(x) = \min \left\{ c_N, \inf_{\substack{e=(x,a,y) \in \mathcal{G} \\ y \in M}} [d_U(e) + \beta z(y)] \right\},$$

where an empty infimum is $+\infty$. Iteration starts from terminal cost C_N . The cap c_N makes every early stop within N steps cost at least C_N after discounting; the block bound gives the same lower bound for a full prefix with its terminal cost. The cap remains available at dead ends, and infinite residuals cannot lower an infimum. Function inequalities and maxima are pointwise.

Theorem 6 (Calibrated finite block construction). *Set*

$$g_0 \equiv C_N, \quad g_j = \mathcal{F}_U^j g_0, \quad \ell_U = \max_{0 \leq j < N} g_j, \quad B = U - \ell_U. \quad (6.10)$$

Then $C_N \leq \ell_U \leq c_N$, and B is a bounded regional Bellman barrier. If specified feasible core histories guarantee the floor $W_M \geq U - \alpha$, with $0 \leq \alpha < C_N$, the staying margin is at least $C_N - \alpha$. Specified target witnesses and independently established adverse comparisons with margin $\varepsilon_{\text{ad}} > 0$ give a uniform certificate with margin $\min\{C_N - \alpha, \varepsilon_{\text{ad}}\}$. For every $0 \leq \alpha < C_N$, both margin bounds are sharp given only $N, \beta, \Delta_{\text{blk}}, \alpha, \varepsilon_{\text{ad}}$.

The construction and sharpness proofs are in Appendix E.1. The construction uses $N - 1$ applications of $\mathcal{F}_U$; with continuous actions, each application still requires bounds on, or solutions to, the action infima.

The guarantees are sharp across problems with the stated summary data; additional information may yield a larger margin. The smaller term determines this guaranteed margin. Stronger block losses can improve staying calibration, a tighter ceiling can improve adverse comparisons, and better witnesses may improve either. Improving only the larger term leaves the guarantee unchanged.

Suppose a staging state has a zero reward self loop and free entry into a target that pays zero forever. Value is zero. The reference $U = 1$ has residual $1 - \beta$ and gives $C_N = 1$, but the smallest admissible calibration error is $\alpha = 1$. Waiting forever is still optimal. Replacing U by $U + u$, for $u \geq 0$, increases every block reference loss by $(1 - \beta^N)u$. The same certified bounds become $C_N + u$ and $\alpha + u$, leaving $C_N - \alpha$ unchanged. Changing the reference level alone therefore cannot create a certified welfare gain.

To use the construction in another problem:

1. Choose staging and target regions for the behavior to be certified.
2. Specify a feasible core plan from every staging state and a plan remaining in the target from every target state. Bound their complete discounted payoffs to obtain witness floors.
3. Establish an independent ceiling at every adverse successor, including all later recovery. An unrestricted Bellman inequality with a valid terminal condition can supply it.
4. Choose a bounded regional reference with nonnegative residuals and bound losses over every feasible staying prefix of a fixed length.
5. Apply Theorem 6, calibrate the barrier against the staging witness, and establish the adverse comparisons separately.
6. Verify the inequalities on the complete feasible graph, including boundary actions, and check terminal conditions.

Weak comparisons give an optimal core replacement whenever an optimum exists; pointwise strict comparisons make every optimizer core. Section 7.2 carries out these steps with two period losses and explicit expansion plans. The regions, plans and ceiling must be supplied for each problem; the plans need not approximate an optimal policy.

6.4 Completeness and the best certificate margins

The converse asks whether a certificate exists whenever every optimizer from the staging or target region is core, and what limits its margin. Call this property *selection*. The result uses exact value under additional assumptions. A *uniform strict* certificate has a common positive margin.

Give states and actions metric topologies. Suppose $M \neq \emptyset$, V is real and attained on $M \cup H$, and $V|_M$ is bounded. At adverse successors, assume V is real or $-\infty$, with every finite value attained. For each feasible edge $e = (x, a, y)$ with $x \in M \cup H$, define exact Bellman loss by

$$d(e) = V(x) - \{r(e) + \beta V(y)\} \in [0, +\infty].$$

Assume the complete staying graph $\mathcal{K} = \{(x, a, y) \in \mathcal{G} : x, y \in M\}$ is compact and d is lower semicontinuous there. For a finite staying path P , put

$$\begin{aligned} \mathcal{L}_n(P) &= \sum_{t < n} \beta^t d(e_t), & \delta_n &= \inf_{|P|=n} \mathcal{L}_n(P), \\ G &= \inf_{\pi \text{ stays in } M} \sum_{t \geq 0} \beta^t d(e_t), & a_* &= \inf_{e \in E_M \cup E_H} d(e). \end{aligned} \tag{6.11}$$

These losses use exact value. The quantities δ_n and G measure the least loss from finite and perpetual staying; a_* measures the least loss on an adverse edge. All initial states in M enter the staying infima. Empty infima are $+\infty$, and $\delta_0 = 0$.

Compactness and lower semicontinuity allow limits of staying paths without increasing loss. Bounded regional value makes discounted terminal values vanish. Attainment at adverse successors allows an optimal continuation after a zero loss departure, which would contradict selection.

Theorem 7 (Compact completeness and sharp margins). *Under these assumptions:*

1. *Every optimizer from $M \cup H$ is core if and only if a pointwise strict FPBB exists. Selection implies a bounded regional barrier with a common positive staying margin. If the adverse graph is also compact and its loss is lower semicontinuous, selection is equivalent to a uniform strict FPBB.*
2. *Suppose selection holds. Let Γ_M be the supremum of uniform staying margins with witness floor $V|_M$, and let Γ be the supremum of full certificate margins over all valid barriers, witness floors, and continuation ceilings, including weak certificates. Then*

$$\Gamma_M = G, \quad \Gamma = \min\{G, a_*\}. \tag{6.12}$$

Every finite staying margin $0 \leq \gamma < G$ has a bounded barrier.

Appendix E.2 turns the absence of an optimal eternal stayer into a positive finite block loss and applies the calibrated construction. Here exact value identifies the suprema across the unrestricted certificate class. The full margin cannot exceed either G or a_* , and attainment of the supremum is not asserted. If $a_* = 0$, every optimum can be core while the full uniform margin is zero.

The labor domain $[0, 1]$ alone does not establish the compact converse assumptions for logarithmic CES utility.

6.5 Contribution and comparison

Bellman inequalities and first exit replacement are established tools (Bellman, 1957; Akao, Ishii, Kamihigashi, and Nishimura, 2022). Gaitsgory, Parkinson, and Shvartsman (2019, Theorem 2.1 and Proposition 3) characterize optimality through normalized Bellman bounds that equal value at the initial state. FPBB uses regional barriers, feasible floors and continuation ceilings without requiring a bound that equals initial value.

Table 1: Two proved limitations and the certificate results

Limitation	Result proved here
Initial welfare accuracy. Extinct histories approach optimal initial welfare uniformly over positive stocks (Proposition 3).	Strict welfare comparisons exclude perpetual staging and every first adverse departure, even when later recovery is allowed, for every optimizer (Theorem 5).
Constant regional barriers. On $M = [.1, .28)$ in (7.3), none can be calibrated against any feasible witness.	A calibrated block barrier and departure comparisons certify finite entry into $H = [.28, \infty)$ and permanence. A separate prefix bound gives the date 46 deadline (Theorem 9).

Lyapunov constructions based on finite iterates yield one period inequalities (Geiselhart, Gielen, Lazar, and Wirth, 2014, Theorems 19 and 20, author version). Finite step control Lyapunov functions support feedback control (Noroozi, Geiselhart, Grüne, and Wirth, 2020). Theorem 6 constructs a bounded barrier from losses over every feasible staying block, calibrates it against feasible plans and gives margins sharp for its stated data. Converse results for discounted storage address robust equilibrium optimality and required supply, assuming reachability from that equilibrium (Grüne, Müller, Kellett, and Weller, 2021, Theorems 5.4 and 6.2). Under compactness, attainment and its other stated hypotheses, Theorem 7 characterizes optimal regional entry and permanence and identifies the sharp margin suprema.

7 An analytical application

Two results connect minimum viable scale to productive entry. The first locates and classifies the frontier on a broad productivity family. The second constructs a first passage certificate on a smaller parameter region and gives an explicit deadline once capital reaches its staging region. On that staging region, every constant regional Bellman barrier fails calibration against every feasible witness.

7.1 An explicit frontier bound

For explicit bounds, specialize the technology to

$$(\theta, \rho, w) = (3/2, -1, 1/2), \quad F_A(K, L) = A \left(\frac{2KL}{K+L} \right)^{3/2},$$

and retain the full labor domain, unrestricted nonnegative capital and logarithmic preferences in (2.1). Consider the family

$$A \geq A_0 := .749, \quad \beta \in [.969, .971], \quad q \in [.919, .921], \quad \eta \in [.699, .701]. \quad (7.1)$$

Theorem 8 (An explicit frontier bound). *For every economy in (7.1), put*

$$\lambda = (A_0/A)^2, \quad b_A = \lambda/20, \quad h_A = 9\lambda/50, \quad \bar{K}_A = 1 + \frac{3A}{.079}.$$

There is a unique minimum viable scale S_A . Every optimum below it becomes extinct, every optimum above it enters $[h_A, \bar{K}_A]$ in finite time and remains there, and attained extinct and

productive optima coexist at S_A . They exhaust the optimum set at that stock. Every extinct optimum from $0 < K \leq S_A$ decreases strictly. The physical boundary and welfare frontier satisfy

$$\frac{(.079)^2}{8A^2} < \kappa_A < S_A < \frac{\lambda}{400} = \frac{.0014025025}{A^2}, \quad 1 < \frac{S_A}{\kappa_A} < 1.798. \quad (7.2)$$

The welfare frontier lies less than 79.8 percent above the lower physical maintenance boundary at every admitted productivity. The threshold .749 is sufficient; the declining upper bound does not establish monotonicity of S_A .

Theorem 2 applies throughout this region. Let $r(K, s)$ be current utility maximized over feasible labor when next capital is s . Appendix F shows that labor has a unique interior global maximizer, establishes $r_{Ks} > 0$ and orders successors on $0 < K \leq 3/5$. The frontier lies strictly inside that interval. Together with classification and the unrestricted foundations, these properties satisfy the geometry proof's hypotheses.

The proof again compares a feasible productive plan with a bound on other histories. Harmonic scaling preserves feasibility at higher productivity by multiplying capital and consumption by λ while retaining labor. The plan beats the normalized welfare bound for histories confined below $\lambda/400$. Local order and exclusion of stationary optima then give the classification and frontier bound. Appendix F contains the proof and finite arithmetic.

7.2 Productive entry and the constant barrier obstruction

For the same harmonic technology, restrict the primitives to

$$A \in [.749, .751], \quad \beta \in [.9699, .9701], \quad q \in [.9199, .9201], \quad \eta \in [.6999, .7001]. \quad (7.3)$$

This box lies inside (7.1); the full labor domain and unrestricted capital remain feasible.

Theorem 9 (Robust CES first passage). *Under (7.3), set $M = [.1, .28]$ and $H = [.28, \infty)$. Every optimal history from M enters H by date 46 and remains there. Every optimal history starting in H remains there. If $K_0 \in [.1, .105]$, its first entry date satisfies*

$$3 \leq \tau_H \leq 46.$$

A bounded regional Bellman barrier and explicit feasible witnesses give a strict first passage certificate with margin .04. No constant regional Bellman barrier can be calibrated below any feasible witness on this same staging region.

The two theorems apply in succession to every productive optimum. For any optimal history with $K_0 > S_A$, let $\tau_{.1} = \inf\{t : K_t \geq .1\}$ and $\tau_H = \inf\{t : K_t \geq .28\}$. Then

$$\tau_{.1} < \infty, \quad \tau_H \leq \tau_{.1} + 46, \quad K_t \in H \quad (t \geq \tau_H). \quad (7.4)$$

Indeed, $h_A \geq (9/50)(.749/.751)^2 > .1$, so Theorem 8 gives the first crossing. Every optimal tail is optimal by Lemma 1. At the crossing it starts in M or directly in H , and Theorem 9 applies. Throughout this box, (7.2) gives $S_A < .0025$; the quantitative certificate starts at .1. No explicit bound is given for the initial delay. Figure 6 displays the two stages.

Following an initial resource loss (Section 4.3), the two-stage guarantee applies whenever retained capital exceeds S_A . After a permanent efficiency loss, this certificate applies only if the resulting primitives remain in (7.3).

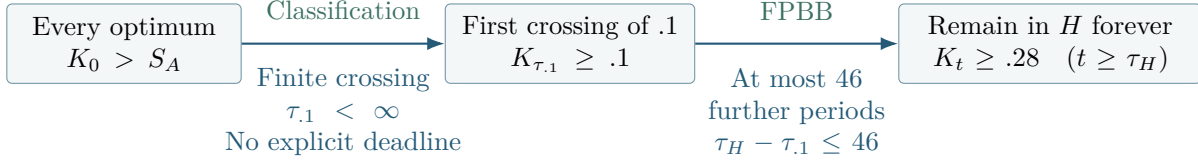


Figure 6: Throughout (7.3), every optimum from $K_0 > S_A$ crosses .1 in finite time; the 46 period bound starts there. Entry may be direct, and crossing dates may coincide or equal zero. There is no explicit initial deadline. Arrows link the guarantees of Theorems 8 and 9.

Write $b = .1$, $h = .28$, and recall the reduced reward $r(K, s)$. The staging and target sets are $M = [b, h)$ and $H = [h, \infty)$. Lemma 12 supplies the independent unrestricted ceiling $\Psi \geq V$ required at adverse successors. We first show why a constant barrier cannot be calibrated on this staging region.

A constant barrier must accommodate the high current reward available near h when next capital is only b . It imposes the same bound at b , where even optimal total welfare is lower. Any constant c satisfying every staying Bellman inequality must obey

$$c \geq \frac{r(h, b)}{1 - \beta} > \frac{-.999}{1 - .9701} > -33.411372,$$

where $r(h, b)$ is approached as current stock tends to h from below. Every feasible witness at b has value at most $V(b) \leq \Psi(b) < -34.27$. No constant barrier can meet the calibration inequality. A successful barrier must therefore vary with the state.

Saving more lowers current consumption and utility. Saving less leaves fewer resources next period. The two period bound captures these complementary losses. Put

$$\hat{R} = -.996, \quad \Delta_2 = .2997, \quad C_{\max} = qh + F_A(h, 1) - b. \quad (7.5)$$

The constant reference $U = \hat{R}/(1 - \beta)$ has nonnegative residual on every staying edge, since its reward is at most $r(h, b) < \hat{R}$. For any two staying edges $x \rightarrow y \rightarrow z$, comparison with the same labor choice at (h, b) gives

$$r(x, y) \leq r(h, b) - \frac{\eta(y - b)}{C_{\max}}, \quad r(y, z) \leq r(h, b) - \frac{\eta q(h - y)}{C_{\max}}.$$

The consumption gains are at least $y - b$ and $q(h - y)$, respectively; the inequalities follow from $\log(c_1/c_0) \geq (c_1 - c_0)/c_1$ and $c_1 \leq C_{\max}$. Since $\beta q < 1$,

$$[\hat{R} - r(x, y)] + \beta[\hat{R} - r(y, z)] \geq \frac{\eta\beta q(h - b)}{C_{\max}} > .2997787 > \Delta_2. \quad (7.6)$$

This comparison covers every feasible staying action, including actions that no optimum would choose.

The two period construction is bounded here without the auxiliary cap: every staging state admits the successor b , and $r(K, b)$ is bounded. Use $N = 2$ and $\Delta_{\text{blk}} = \Delta_2$. Let $\mathcal{T}_M$ be the Bellman operator restricted to staying successors and set

$$U = \frac{\hat{R}}{1 - \beta}, \quad B_0 = U - \frac{\Delta_2}{1 - \beta^2}, \quad B(K) = \min\{B_0, r(K, b) + \beta B_0\}. \quad (7.7)$$

To verify the barrier directly, (7.6) implies $\mathcal{T}_M^2 B_0 \leq B_0$, while $\mathcal{T}_M B_0(K) = r(K, b) + \beta B_0$. Monotonicity gives

$$\mathcal{T}_M B \leq \min\{\mathcal{T}_M B_0, \mathcal{T}_M^2 B_0\} \leq \min\{\mathcal{T}_M B_0, B_0\} = B.$$

The barrier is bounded and satisfies $B \leq B_0 \leq -38.144176$. The staging witness chooses labor .68 and next capital .135, then follows the expansion path in Table 4 and eventually maintains capital at 2.830. Its complete continuation gives $W_M > -35.58$, so the staying margin exceeds 2.564. Appendix F.3 verifies feasibility and the payoff bound throughout M .

The target floor W_H comes from two plans. For stocks in $[h, .8]$, the first chooses labor .72 and next capital .375; for stocks in $[.8, \infty)$, the second chooses labor .78 and next capital .9. Each then follows its corresponding tail in Table 4 and remains in H . Appendix F.3 bounds their complete payoffs throughout the target region.

Departure comparisons use the unrestricted ceiling $\Psi^\# = \Psi - 10/301 \geq V$ from Lemma 12. Appendix F.2 verifies a curved bound, a global affine bound and their join, together with the terminal condition. Only the low stock branch is needed at adverse successors:

$$\Psi^\#(s) = \psi(\max\{s, .05\}) - \frac{10}{301}, \quad \psi(s) = .4s - 5s^{-1/2} - 18.5, \quad 0 \leq s \leq h. \quad (7.8)$$

For every staging departure below b and target departure below h , the appendix places current utility plus unrestricted continuation more than .04 below the corresponding witness. The comparison covers the full labor interval and every subsequent recovery path. Theorem 5 makes every optimizer core. The reported .04 is a conservative guarantee governed by adverse comparisons; the staying margin exceeds 2.564. This certificate does not require the compact converse assumptions.

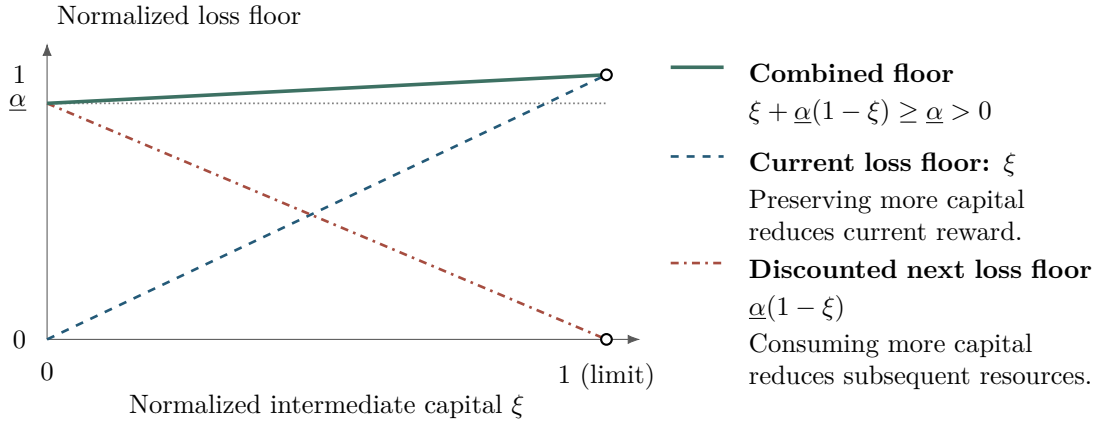
A separate finite prefix comparison gives the deadline. Accumulating 23 two period losses and adding the unrestricted continuation ceiling at date 46 places welfare from 46 consecutive staying transitions below the staging witness. No optimum can delay entry beyond date 46. Physical resource bounds give the lower entry date of three from $[.1, .105]$. The appendix supplies both calculations.

Figure 7 separates the block mechanism from the calibration obstruction. For two staying transitions $x \rightarrow y \rightarrow s$, write

$$\xi = \frac{y - b}{h - b} \in [0, 1), \quad d_1 = \hat{R} - r(x, y), \quad d_2 = \hat{R} - r(y, s), \quad a = \frac{\eta(h - b)}{C_{\max}}.$$

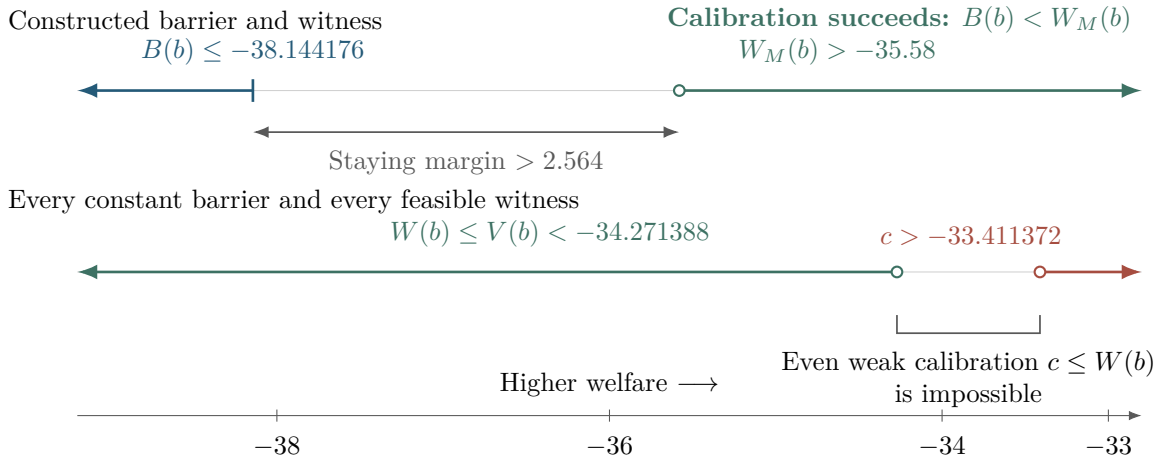
Let β_-, q_- be the lower endpoints in (7.3). With $\underline{\alpha} = \beta_- q_- = .89221101$, the inequalities above give $d_1/a \geq \xi$ and $\beta d_2/a \geq \underline{\alpha}(1 - \xi)$. Their sum is at least $\xi + \underline{\alpha}(1 - \xi) \geq \underline{\alpha}$.

(a) Complementary losses over two periods



The block loss lowers the constructed barrier below a feasible witness's payoff.

(b) A calibrated barrier and an impossible constant calibration



Shared welfare scale; markers denote bounds, not realized values.

Full certificate: margin .04, governed by adverse comparisons.

Figure 7: Block losses and calibration throughout (7.3). Panel (a) plots normalized loss floors, with the next loss discounted; actual losses for every feasible labor choice lie above them. Zero floors need not mean zero losses, and not every intermediate stock is feasible from every origin. Panel (b) shows bounds at $b = .1$. Every constant regional barrier fails even weak calibration against every feasible witness; the constructed barrier satisfies a one period Bellman inequality.

Classification determines which optimal paths survive; constructive verification bounds their subsequent productive entry and permanence without an exact value function.

A Foundations and classification

A.1 Foundations and the welfare bound

Maintain (2.1): labor lies in $[0, 1]$, the primitive vector is $p = (A, \beta, q, \eta, \theta, \rho, w)$, and $1 < \theta < 2$. Utility is $-\infty$ at $C = 0$ or $L = 1$. With the capacity coefficient H_a in (2.4), put

$$R_p = q + AH_a, \quad a_p = \frac{\eta}{1 - \beta}, \quad \mathcal{D}_p = \frac{\eta}{(1 - \beta)^2} \log \frac{R_p}{q}. \quad (\text{A.1})$$

Lemma 1 (Economic foundations). *At every primitive vector in (2.1), value is finite and continuous on $(0, \infty)$, and optimal histories exist. The optimal history and successor correspondences have compact values and closed graphs locally on that domain. Bellman's equation holds, and every optimal tail is optimal. Every optimal action exhausts resources and has positive consumption, positive successor capital and $L < 1$. On compact positive initial stock intervals, optimal first consumption, successor and leisure are uniformly bounded away from zero. If optimal stocks converge to $k > 0$, then $k \in G_p(k)$.*

Proof. Output is at most $P = A(1 - w)^{\theta/\rho}$. Every feasible history from K stays below $\max\{K, P/(1 - q)\}$, and a common stock bound works on compact initial stock intervals. With consumption retained as a coordinate, the feasible history set is closed in a common compact product of intervals. Extend utility upper semicontinuously at zero consumption and zero leisure. If M_u bounds current utility above, welfare equals $M_u/(1 - \beta)$ less the sum of nonnegative discounted losses $M_u - u_p$. Fatou's lemma gives upper semicontinuity even at welfare $-\infty$. Cake supplies the finite welfare (5.1), so the maximum is finite and attained. Zero consumption, full labor or a zero successor would give welfare $-\infty$; a zero successor forces zero consumption later. Any discarded resources could raise first consumption, so every optimum exhausts resources.

Limits of optimizers on a common compact cover give upper semicontinuity of value. For lower semicontinuity, keep an optimizer's first labor, successor and entire tail fixed. Positive first consumption absorbs small changes in initial resources. With labor strictly below one, welfare varies continuously under this adjustment. Hence value is continuous. Compactness also gives closed optimal history graphs and compact optimal sets; their projections give the successor claims. Over a compact positive initial stock interval, the graph of optimal histories is compact in a common product cover. Its first consumption, successor and leisure are positive continuous coordinates, so their minima are positive. Splitting and concatenating optimal histories proves Bellman's equation and optimality of tails. Finally, closedness of the optimal successor graph applied to $(K_t, K_{t+1}) \rightarrow (k, k)$ gives $k \in G_p(k)$. $\square$

Lemma 2 (Cake normalization and its terminal term). *On positive consumption and successor edges define $r_p^0 = u_p - E_p(K) + \beta E_p(K')$. For a history π from K_0 , write $J_{K_0}^0(\pi) = \sum_{t \geq 0} \beta^t r_p^0(e_t)$ for normalized welfare. Among histories with positive consumption and successor capital, original welfare is finite if and only if normalized welfare is finite. On these finite welfare histories, $\beta^t E_p(K_t) \rightarrow 0$ and $J_{K_0}^0(\pi) = J_{K_0}(\pi) - E_p(K_0)$. An edge with $L = 1$ has reward $-\infty$ in both formulations.*

Proof. The capacity bound gives $F_p(K, L)/K \leq AH_a L^{\theta-1}$ and resources at most $R_p K$. Put $s = K'$ for successor capital. For positive C, s , write $Q = qK + F_p(K, L)$ and $z = s/Q$. Direct subtraction gives

$$\begin{aligned} u_p(C, L) - E_p(K) + \beta E_p(s) &= \frac{\eta}{1-\beta} \left\{ \log \frac{Q}{qK} - D_\beta(z) \right\} + (1-\eta) \log(1-L) \\ &\quad + \eta \log \frac{C}{Q-s}, \\ D_\beta(z) &= (1-\beta) \log \frac{1-\beta}{1-z} + \beta \log \frac{\beta}{z} \geq 0. \end{aligned} \tag{A.2}$$

The last inequality follows by differentiation; equality holds at $z = \beta$. The identity bounds every normalized reward above by $a_p \log(R_p/q)$. Original rewards are bounded above on the history's resource cover. Both discounted sums are therefore well defined.

For a history with finite original welfare, bound $\log C_t$ above by M_C on its resource cover. The losses $\eta(M_C - \log C_t) + (1-\eta)[- \log(1-L_t)]$ form a summable nonnegative discounted series. Each component is nonnegative, so $\beta^t \log C_t \rightarrow 0$. The inequality $\log C_t - \log R_p \leq \log K_t$, together with the finite upper stock cover, gives $\beta^t E_p(K_t) \rightarrow 0$. For each finite positive prefix,

$$\sum_{t < T} \beta^t r_p^0(e_t) = \sum_{t < T} \beta^t u_p(e_t) - E_p(K_0) + \beta^T E_p(K_T). \tag{A.3}$$

This gives the identity when original welfare is finite. If original welfare is $-\infty$, its partial sums tend to $-\infty$, while $E_p(K_T)$ is bounded above on the stock cover. The normalized sum is then $-\infty$ as well. Since original welfare cannot be $+\infty$, the two formulations have the same finite welfare histories. This establishes equivalence without subtracting infinite payoffs. An edge using full labor makes both sums $-\infty$. $\square$

Proof of Proposition 3. The preceding identity gives the uniform stage bound $r_p^0 \leq a_p \log(R_p/q)$. For histories with finite welfare, the terminal identity therefore yields $J_{K_0}(\pi) - E_p(K_0) \leq \mathcal{D}_p$. The bound also holds at welfare $-\infty$, including zero consumption, full labor and zero successors. Cake gives the lower bound. Thus

$$0 \leq V_p(K) - E_p(K) \leq \mathcal{D}_p.$$

Lemma 1 supplies attainment and optimal tails. Replacing an optimal tail at date n by cake gives exactly (6.2); its loss is at most $\beta^n \mathcal{D}_p$. Letting $n \rightarrow \infty$ proves equality of the extinct and unrestricted welfare suprema.

Uniform interiority of relevant Bellman actions. On a compact positive initial stock interval, every Bellman action whose objective is bounded below by a common finite constant has consumption, successor and leisure bounded away from zero. Current resources and continuation welfare have common upper bounds. As consumption or leisure approaches zero, the objective therefore tends to $-\infty$. A successor approaching zero has the same effect because $V_p(s) \leq E_p(s) + \mathcal{D}_p \rightarrow -\infty$. The argument applies to both restricted and unrestricted Bellman problems. $\square$

A.2 Optimal action bounds and productive quantities

The resource constant $R_p = q + AH_a$ and the welfare bound $\mathcal{D}_p$ in (6.1) define

$$\chi_p = (1 - \beta)q \left(\frac{\beta q}{R_p} \right)^{\beta/(1-\beta)} \exp(-\beta \mathcal{D}_p/\eta), \quad 0 < \chi_p < q. \quad (\text{A.4})$$

Proposition 4 (Consumption, leisure and productive activity). *At every primitive vector in (2.1) and every positive stock K , each optimal action obeys*

$$C(1 - L)^\nu \geq \chi_p K, \quad C \geq \chi_p K, \quad 1 - L \geq (\chi_p/R_p)^{1/\nu} > 0. \quad (\text{A.5})$$

If an optimal tail remains in $[h, \bar{K}]$, where $0 < h \leq \bar{K} < \infty$, put $c = \chi_p h$, $d_0 = (q - \chi_p)(\bar{K} - h)$ and $f_0 = (1 - q)h + c$. Consumption on that tail is at least $c > 0$, and every window of $T \geq 1$ dates satisfies

$$\frac{1}{T} \sum_{j=0}^{T-1} F_p(K_{t+j}, L_{t+j}) \geq f_0 - \frac{d_0}{T}. \quad (\text{A.6})$$

Proof of Proposition 4 and Corollary 1. For an optimal action with positive successor s , Bellman's equation and the independently established cake bounds give

$$\begin{aligned} \eta \log[C(1 - L)^\nu] &= V_p(K) - \beta V_p(s) \\ &\geq E_p(K) - \beta E_p(s) - \beta \mathcal{D}_p \\ &\geq E_p(K) - \beta E_p(R_p K) - \beta \mathcal{D}_p. \end{aligned}$$

The last inequality uses $s \leq R_p K$ and the fact that cake value increases with capital. Expanding (5.1) and exponentiating gives $C(1 - L)^\nu \geq \chi_p K$. Since $1 - L \leq 1$, this implies $C \geq \chi_p K$; since $C \leq R_p K$, it also gives the leisure floor. Formula (A.4) gives $0 < \chi_p < q$. These restrictions apply to optimal actions only; the barrier comparisons still cover the complete feasible graph.

On an optimal tail in $[h, \bar{K}]$, resource exhaustion gives

$$\begin{aligned} \sum_{j=0}^{T-1} F_p(K_{t+j}, L_{t+j}) &\geq (\chi_p - q)K_t + (1 - q + \chi_p) \sum_{j=1}^{T-1} K_{t+j} + K_{t+T} \\ &\geq (1 - q + \chi_p)hT - (q - \chi_p)(\bar{K} - h). \end{aligned}$$

This proves the window bound, including $T = 1$.

Finally every optimal transition satisfies $K' \leq R_p K - C \leq (R_p - \chi_p)K$. If $A \leq A_{\text{rep}}$, then $R_p \leq 1$ and $0 < q - \chi_p < R_p - \chi_p < 1$. Thus the explicit contraction factor is

$$\gamma_p = R_p - \chi_p \in (0, 1). \quad (\text{A.7})$$

Iteration proves (2.7), independently of classification or productive capture. $\square$

A.3 Physical vulnerability to efficiency losses

Proof of Proposition 2. The signs in (4.2) give the nesting in (4.12). At productivity $A_e > A_{\text{rep}}$, write $\Phi_e(K) = qK + A_e y(K)$. Below $\kappa(A_e)$ one has $\Phi_e(K) < K$. On any $[0, K_1]$ with $0 < K_1 < \kappa(A_e)$,

the ratio $\Phi_e(K)/K$, extended by q at zero, has a maximum strictly below one. Every feasible history starting there therefore contracts to zero. At $\kappa(A_e)$, positive first consumption sends the successor strictly below the root. Every finite welfare history has positive consumption, so extinction follows throughout the stated band. By (4.12), that band lies strictly inside the original sustainable interval.

If $\tau > \tau^*$, effective productivity is below A_{rep} , and Proposition 1 gives the global contraction bound. At equality, Corollary 1 gives optimal extinction. The maintenance path at k_a remains feasible but has welfare $-\infty$. In each extinction case, $F_{A_e}(K_t, L_t) \leq A_e w^{\theta/\rho} K_t^\theta \rightarrow 0$. $\square$

A.4 Local order and the frontier

Lemma 3 (An ordered frontier between two anchors). *Suppose the conclusions of Lemma 1 hold. Let $0 < a < b < h$. Suppose every optimum from $(0, a]$ is extinct, while every optimum from $[b, \infty)$ enters and remains in a fixed compact target contained in $[h, \infty)$. Suppose also that*

$$\max G(x) \leq \min G(y) \quad (a \leq x < y \leq b), \quad x \notin G(x) \quad (a \leq x \leq b).$$

There is a unique $S \in (a, b)$ with unanimous extinction below, unanimous productive survival above, and attained coexistence at S . Within the bracket, extinct successors are strictly below their current stock, productive successors strictly above it, and the optimal successors from S lie on both sides and never equal S .

Proof. Append an optimal tail to any finite chain of Bellman maximizing actions. For an infinite maximizing chain, extend each finite prefix in this way and take a convergent subsequence in the compact optimal history set. Every fixed prefix eventually agrees with the prescribed chain, so the limit is that chain and is optimal. Least and greatest successor selections therefore generate optima, without an additional terminal condition for the chain.

Direction and finite exit. Suppose $x < y \in G(x)$ and both stocks are in $[a, b]$. Order gives $y \leq \min G(y)$, and exclusion of fixed optimal transitions makes the inequality strict. A downward move likewise forces all subsequent moves downward while the history remains in the bracket. Any optimal history confined to $[a, b]$ would therefore be strictly monotone and converge to $k \in [a, b]$. Closedness applied to successive stocks would imply $k \in G(k)$, a contradiction. Every optimum exits the bracket in finite time, toward the anchor corresponding to its first move. An upward optimum from a would reach the productive anchor, contradicting extinction there. A downward optimum from b would reach the extinct anchor, contradicting productive capture. Thus all successors at a are below a , and all at b are above b .

An upward history cannot start below a downward history. Suppose an upward optimum starts at x_0 below a downward optimum starting at y_0 , both inside the bracket. As long as $x_t < y_t$, order gives $x_{t+1} \leq y_{t+1}$. Equality at a stock z is impossible: direction gives $x_t < z < y_t$, while order between these three origins forces $G(z) = \{z\}$. Thus the inequality remains strict. Both histories are trapped in $[x_0, y_0]$: the lower one increases and cannot cross the decreasing upper one. Neither can exit the bracket first, since an upward exit would cross the upper history and a downward exit would cross the lower history. This contradicts finite exit.

The frontier and attainment. Consequently the states with only downward successors form a down set, and those with only upward successors form an up set. They are nonempty at a and b and relatively open by upper hemicontinuity. Every other state has successors on both sides, since its compact successor set omits the state itself. There is at most one such state: two would give an upward optimum below a downward optimum. The two disjoint, nonempty, relatively

open sets cannot cover the connected interval $[a, b]$. Hence there is a remaining state, denoted $S \in (a, b)$, and the preceding argument makes it unique. Every downward choice at S , followed by an optimal tail, yields an attained extinct optimum; every upward choice yields an attained productive optimum. The anchors and direction argument classify each tail, and the two classes exhaust the optimum set. $\square$

A.5 The productive anchor and local choice order

The construction below supplies $0 < b < h < 1 < \bar{K}(A)$, a feasible productive plan and an unrestricted welfare bound after every low successor. Let $\Phi(K) = qK + F_p(K, 1)$ and

$$R_{\text{cap}} = q\bar{K}(A) + A(1 - w)^{\theta/\rho}.$$

The construction gives $\Phi(\bar{K}(A)) < R_{\text{cap}} < \bar{K}(A)$, so the resource cover $[0, \bar{K}(A)]$ is invariant.

Set $\sigma = 3/4$ and $Z(K) = qK + F_p(K, \sigma)$. A feasible witness uses labor σ , sets successor capital to one and holds it there. Its welfare is

$$\begin{aligned} W(K) &= \eta \log[Z(K) - 1] + (1 - \eta) \log(1 - \sigma) \\ &\quad + \frac{\beta}{1 - \beta} \{ \eta \log[F_p(1, \sigma) - \delta] + (1 - \eta) \log(1 - \sigma) \}. \end{aligned} \quad (\text{A.8})$$

The productivity conditions make consumption positive at b and after the reset. Increasing resources preserve feasibility for every $K \geq b$. The reset stock lies above h .

For unrestricted continuation on the cover, use

$$\bar{V}(y) = \eta \log \Phi(y) + \frac{\beta \eta}{1 - \beta} \log R_{\text{cap}}, \quad \bar{V}(0) = -\infty. \quad (\text{A.9})$$

Current consumption is at most $\Phi(y)$, every later consumption is at most R_{cap} , and leisure utility is nonpositive. The ceiling therefore bounds every continuation. It is conservative: it ignores labor costs and permits consumption bounds at different dates to be attained independently.

Primitive verification constants

Fix the six primitives other than A and assume (2.3). First bound the stationary candidates, then choose the resource cover and the capture stocks b, h , and finally set the productivity threshold. The capture stocks and threshold do not depend on A ; the resource cover grows with it. Low-stock extinction, productive capture, local successor order and exclusion of fixed optimal transitions then supply the hypotheses of Lemma 3.

Stationary bounds. Put $\alpha = 1/\nu$, $\xi = -1/\rho$, $c_w = (1 - w)/w$, and

$$z_C = \frac{(\theta - 1)\xi}{1 + \xi}, \quad L_* = \frac{\theta}{\theta + \nu}.$$

The proof of Lemma 6 parameterizes the stationary candidates by $z > z_m := -E/D > 0$:

$$\begin{aligned} L(z) &= \frac{\theta D z}{\nu E + (\theta + \nu) D z}, \quad K(z) = c_w^{-\xi} z^\xi L(z), \\ \mathcal{A}(z) &= \frac{D(1 - w)^{\theta\xi} c_w^{-\xi} (1 + z)^{1 + \theta\xi}}{\theta L(z)^{\theta-1} z^{(\theta-1)\xi}}. \end{aligned} \quad (\text{A.10})$$

Here $L_* < L(z) < 1$. Put $z_u = \max\{z_m, z_C\}$ and

$$k_* = c_w^{-\xi} L_* z_u^\xi, \quad A_s = \frac{D(1-w)^{\theta\xi} c_w^{-\xi} (1+z_u)^{1+\theta\xi}}{\theta L_*^{\theta-1} z_m^{(\theta-1)\xi}}. \quad (\text{A.11})$$

Witness and resource bounds. Fix witness labor $\sigma = 3/4$ while retaining the full feasible labor domain. Distinguish witness output from physical capacity:

$$f(K) = \{wK^\rho + (1-w)\sigma^\rho\}^{\theta/\rho}, \quad f_1(K) = \{wK^\rho + 1-w\}^{\theta/\rho}, \\ M_0 = (1-w)^{\theta/\rho}, \quad \gamma > M_0/\delta, \quad Q = q\gamma + M_0.$$

Target regions. Choose the cover coefficient first; $\gamma = 1 + 2M_0/\delta$ is admissible. Define

$$h_* = \min\{1/2, k_*/2, 1/9, 2k_a\}, \\ T = \exp \max \left\{ \log \frac{f(1)}{f(h_*)}, \right. \\ \left. \frac{1 - \nu \log(1 - \sigma) - (1 - \beta^2)\theta \log \sigma + \beta^2 \log[Q/f(1)]}{\beta(1 - \beta)} \right\}, \quad (\text{A.12}) \\ h = \left[\frac{(f(1)/T)^{\rho/\theta} - (1-w)\sigma^\rho}{w} \right]^{1/\rho}, \\ b = h/2, \quad \bar{K}(A) = \gamma A, \quad t = \frac{e^{1/2} - 1}{e^{1/2} + 1}.$$

The definition of h gives $f(h) = f(1)/T$. The first term defining T and strict increase of f therefore ensure $h \leq h_*$. Consequently $0 < b < h \leq h_* < 1$, $h < k_*$, $9b < 1$, and $b \leq k_a$. Since $\bar{K}(A) = \gamma A$, one has $R_{\text{cap}} = AQ$.

Productivity gates. The productivity threshold below has five terms. In order, they ensure witness feasibility and the strict welfare margin; exclusion of stationary optima below the target; maintenance at $b/2$, which places the lower physical root below the capture region; positivity of the reduced payoff's cross derivative; and global domination of zero labor. The last two terms establish local successor order over the full labor choice set. Put

$$j(u) = \theta - 1 - (\theta - \rho) \frac{w}{w + (1-w)u^\rho}, \\ t_\varphi = \left[\frac{w(1-\rho)}{(1-w)(\theta-1)} \right]^{1/\rho}, \quad j_* = j(t_\varphi/2) > 0, \\ c = \min \left\{ t_\varphi/2, \frac{\alpha j_*}{b[2(1+\alpha) + \alpha j_*]} \right\}, \\ f_o(u) = \frac{\theta w}{u} \{w + (1-w)u^\rho\}^{\theta/\rho-1}, \quad (\text{A.13}) \\ A_* = \max \left\{ \frac{1}{tf(b)}, A_s, \frac{\delta b}{2f_1(b/2)}, \right. \\ \left. \frac{qb^{2-\theta}}{\alpha f_o(c)}, \frac{qb^{1-\theta}[(1-9b)^{-\nu} - 1]}{[w + (1-w)9^\rho]^{\theta/\rho}} \right\}.$$

Once the cover coefficient is fixed, these finite constants depend only on the six primitives other than A . Fix $A > A_*$ and write $Z(K) = qK + Af(K)$ and $\Phi(K) = qK + Af_1(K)$.

Lemma 4 (Productive anchor and capture). *Under (A.12)–(A.13), every optimal successor from $K \geq b$ is at least h . Every optimum from that domain eventually enters $[h, \bar{K}(A)]$ and remains there.*

Proof. Witness and unrestricted ceiling. Output at full labor is below AM_0 , so $[0, \bar{K}(A)]$ is invariant and $\Phi(\bar{K}(A)) < AQ < \bar{K}(A)$. Also $tf(b) < M_0 < \gamma$ and $A > 1/[tf(b)]$ give $1 < \bar{K}(A)$. Labor σ , resetting capital to one and then maintaining it there, earns (A.8). For $K \geq b$, one has $Z(K) > 1/t$ and hence $Z(K) - 1 > (1 - t)Z(K)$. Consumption after the reset exceeds $(1 - t)Af(1)$, so the witness is feasible throughout the domain. Every unrestricted continuation from $0 < y \leq \bar{K}(A)$ has the independent ceiling (A.9): first consumption is at most $\Phi(y)$, and invariance bounds every later consumption by AQ . This ceiling uses the full resource bound and nonpositive leisure utility; it allows recovery after a low successor.

Excluding low successors. Homogeneity and monotonicity give $f(K) = \sigma^\theta f_1(K/\sigma) \geq \sigma^\theta f_1(K)$. Hence $\Phi(K) \leq \sigma^{-\theta} Z(K)$ at every positive stock. Moreover $Z(h) \leq (1 + t)Af(h)$, since $h < 1$ and $1 < tAf(b)$. For $K \geq b$ and $0 < y \leq h$, current reward and the unrestricted continuation therefore have the bound

$$u_p(C, L) + \beta \bar{V}(y) \leq \eta \log Z(K) + \beta \eta \log Z(h) - (1 + \beta) \eta \theta \log \sigma + \frac{\beta^2 \eta}{1 - \beta} \log(AQ).$$

The bound covers every $L \in [0, 1]$, with value $-\infty$ at full labor. The advantage of the witness $W(K)$ over this upper bound, multiplied by $(1 - \beta)/\eta$, is at least

$$\begin{aligned} & \beta(1 - \beta) \log \frac{f(1)}{f(h)} - \beta^2 \log \frac{Q}{f(1)} + \nu \log(1 - \sigma) + (1 - \beta^2) \theta \log \sigma \\ & + \log(1 - t) - \beta(1 - \beta) \log(1 + t) \geq \frac{1}{2}. \end{aligned}$$

The first line is at least one by (A.12); the last two terms are at least $\log[(1 - t)/(1 + t)] = -1/2$.

Immediate entry and capture. A zero successor has value $-\infty$. The strict comparison just proved therefore excludes every optimal successor at or below h from all $K \geq b$, including stocks above the cover. Optimal tails imply permanence above h : every later stock remains in the domain of the same comparison. Invariance therefore puts every optimal successor from the staging region $M = [b, h]$ in $[h, \bar{K}(A)]$. For larger initial stocks, the resource bound gives $K_n \leq q^n(K_0 - AM_0/\delta) + AM_0/\delta$. Since $\bar{K}(A) > AM_0/\delta$, every such history enters the cover in finite time and then remains there. This proves productive capture. $\square$

Lemma 5 (Global labor choice and local successor order). *Under the same construction, for $0 < K \leq b$, labor maximization at any successor permitting positive consumption has a unique solution in $(0, 1)$. After labor is maximized, the period payoff has strict increasing differences on feasible crossed rectangles in this stock domain, and $\max G_p(x) \leq \min G_p(y)$ for $0 < x < y \leq b$.*

Proof. For $0 < K \leq b$, replace any feasible zero labor choice by $9K$, keeping its successor fixed. The gain is at least

$$J(K) = \eta \log(1 + d_9 K^{\theta-1}) + (1 - \eta) \log(1 - 9K), \quad d_9 = \frac{A}{q} [w + (1 - w)9^\rho]^{\theta/\rho}.$$

Its derivative has the sign of

$$\eta d_9(\theta - 1) - 9(1 - \eta)K^{2-\theta} - 9d_9[1 - \eta + \eta(\theta - 1)]K,$$

a strictly decreasing expression. Thus the gain can first rise and then fall, but cannot have an interior minimum. The productivity bound for domination of zero labor in (A.13) gives $J(b) > 0$; $J(0) = 0$ then implies $J(K) > 0$ on $(0, b]$. Thus zero labor is either infeasible or strictly suboptimal. Full labor has utility $-\infty$.

At fixed K , define $\zeta(L) = F_p - \alpha(1 - L)F_{p,L}$ and $T_L = (1 + \alpha)F_{p,L} - \alpha(1 - L)F_{p,LL}$. For positive consumption, the labor derivative at fixed successor is

$$\frac{\partial}{\partial L} u_p(qK + F_p(K, L) - s, L) = \frac{1 - \eta}{C(1 - L)}[s - qK - \zeta(L)].$$

Thus the interior labor equation is $s - qK = \zeta(L)$. The sign of $T_L = \zeta'(L)$ is the sign of $(1 + \alpha)L/(1 - L) - \alpha j(L/K)$, a strictly increasing expression. It is negative near zero and tends to infinity as $L \uparrow 1$. Thus ζ decreases and then increases. A solution of the labor equation on the decreasing branch of ζ changes the utility derivative from negative to positive and can only be a local minimum. The increasing branch has at most one solution, where the derivative changes from positive to negative. At the turning point itself, a tangency does not create a maximum. The feasible positive consumption labor set is an interval; utility tends to $-\infty$ at a positive lower labor boundary where consumption vanishes, and at full labor. Since zero labor is infeasible or dominated, the global maximum is unique, interior, and has $T_L > 0$. The choice of c gives

$$\frac{(1 + \alpha)cb}{1 - cb} \leq \frac{\alpha j_*}{2} < \alpha j(c),$$

which, since $K \leq b$, implies $T_L < 0$ at $L = cK$. The unique interior maximizer lies on the increasing branch of ζ , so its labor exceeds cK .

Let $r_p(K, s)$ denote the period payoff after maximizing over labor. Strict increasing differences means that the gain from a larger successor rises with current capital whenever the four compared choices are feasible. Implicit differentiation on the maximizing branch gives

$$r_{p,Ks} = \frac{\eta\alpha}{C^2 T_L} [(q + F_{p,K})\{F_{p,L} - (1 - L)F_{p,LL}\} + (1 - L)F_{p,L}F_{p,KL}].$$

Substituting the CES derivatives reduces this expression to

$$\begin{aligned} r_{p,Ks} &= \frac{\eta\alpha\mathcal{N}}{C^2 T_L}, \quad \mathcal{N} = \frac{F_{p,L}}{L} \{qn + F_{p,K}[1 - \rho(1 - L)]\}, \\ n &= L - (1 - L)j(L/K) > -L/\alpha. \end{aligned} \tag{A.14}$$

If $n \geq 0$, then $\mathcal{N} > 0$. If $n < 0$, then $j(L/K) > 0$. On the interval where $j(u) > 0$, $f_o(u)$ increases: its logarithmic derivative times u is $(\theta - \rho)[1 - w/(w + (1 - w)u^\rho)] - 1 > -\rho > 0$. The function j decreases, so its positive set is an interval containing both c and L/K . Since $L/K > c$, the productivity bound $A > qb^{2-\theta}/[\alpha f_o(c)]$ gives $F_{p,K}/L = AK^{\theta-2}f_o(L/K) > q/\alpha$. Consequently $\mathcal{N} > 0$ again. Integrating across a feasible crossed rectangle gives strict increasing differences.

Suppose $0 < x < y \leq b$ had optimal successors $s_x > s_y$. Since s_x is feasible with positive consumption at x , every successor in $[s_y, s_x]$ is feasible at every stock in $[x, y]$: resources increase with capital and a smaller successor releases consumption. Bellman optimality would give

$$\begin{aligned} r_p(x, s_x) + \beta V_p(s_x) &\geq r_p(x, s_y) + \beta V_p(s_y), \\ r_p(y, s_y) + \beta V_p(s_y) &\geq r_p(y, s_x) + \beta V_p(s_x). \end{aligned}$$

Cancelling the continuation values contradicts strict increasing differences on that rectangle. Therefore

$$\max G_p(x) \leq \min G_p(y) \quad (0 < x < y \leq b). \quad (\text{A.15})$$

□

Lemma 6 (Exclusion of low stationary optima). *Under the same construction, $K \notin G_p(K)$ for every $0 < K \leq b$.*

Proof. Bellman's equation allows a fixed optimal transition to be repeated at a stationary action (C, L, K) . Keep labor at its stationary level and vary the date one stock to $K + v$, restoring capital to K at date two. Consumption becomes $C - v$ initially and $q(K + v) + F_p(K + v, L) - K$ one date later. Both signs of sufficiently small v are feasible. The derivative of welfare at zero is

$$-\frac{\eta}{C} + \beta \frac{\eta}{C} (q + F_{p,K}) = 0,$$

so $F_{p,K} = D$. This necessary condition uses only a feasible variation over two dates; no concavity or Euler sufficiency is used.

Positive consumption requires $L > 0$, and finite welfare requires $L < 1$, so the labor condition is interior. Put $z = c_w(K/L)^{-\rho}$. The factor identities are

$$KF_{p,K} = \frac{\theta F_p}{1 + z}, \quad LF_{p,L} = \frac{\theta F_p z}{1 + z}.$$

Using $F_{p,K} = D$ and $C = F_p - \delta K$ gives $C/K = (E + Dz)/\theta$. Thus $z > z_m$. Substitution into $\nu C = (1 - L)F_{p,L}$ gives $L = \theta Dz / [\nu E + (\theta + \nu)Dz]$. The definition of z then gives $K(z)$, and the production identity gives $\mathcal{A}(z) = A$ in (A.10).

Since $E < 0$, every candidate has $L(z) > L_*$. On $(z_m, z_u]$, (A.11) bounds $\mathcal{A}(z)$ above by A_s , excluding every root when $A > A_s$. For $z > z_u$, $K(z) > k_* > h$. Thus no fixed optimal transition exists at $0 < K \leq b$. □

Proof of Theorem 1. Physical extinction and the frontier. Full capacity average product $f_1(K)/K$ increases until k_a and then decreases to zero. The maintenance bound $A > \delta b / [2f_1(b/2)]$ gives $Af_1(b/2) > \delta b/2$; since $b \leq k_a$, there are two physical replacement roots and the lower one satisfies $\kappa_p < b/2$. Below that root, $\Phi(K) < K$. Positive first consumption at the root sends capital strictly below it. Once a finite welfare history reaches $0 < K_1 < \kappa_p$, the continuous ratio $\Phi(K)/K$, extended by $q < 1$ at zero, has a maximum strictly below one on $[0, K_1]$. Such a history contracts to zero. Every optimum from $K \leq \kappa_p$ is therefore extinct.

Apply Lemma 3 with lower anchor κ_p , upper anchor b and the productive target already established. It gives the unique frontier, attained coexistence and successor directions. An extinct optimum from the frontier first moves below it. Order forces strict decreases until the physical extinction region, where the resource constraint does so thereafter. Proposition 1 gives constant maintenance with positive consumption and leisure on $(\kappa_p, b]$.

Primitive neighborhoods and frontier continuity. The inequalities establishing patience, the five verification gates and the exclusion of stationary optima are strict. Their bounds depend continuously on the six primitives other than A . The lower physical root is simple. Classification therefore persists on an open neighborhood in all seven primitives, with a common resource cover after shrinking the neighborhood.

Frontier continuity also requires control of optimal choices as primitives vary. Take a compact primitive neighborhood and a common invariant cover $[0, \bar{K}]$. Write $K = \bar{K}k$ and

$$g_p(k, L) = q + \frac{F_p(\bar{K}k, L)}{\bar{K}k} \quad (k > 0), \quad g_p(0, L) = q.$$

The bound $0 \leq g_p - q \leq Aw^{\theta/\rho}(\bar{K}k)^{\theta-1}$ gives joint continuity at zero, uniformly in labor and primitives on that neighborhood. The normalized Bellman operator is

$$(\mathcal{T}_p\varphi)(k) = \sup_{0 \leq L \leq 1, 0 < z < 1} \left\{ \frac{\eta}{1-\beta} [\log(g_p/q) - D_\beta(z)] + (1-\eta) \log(1-L) + \beta\varphi(kg_pz) \right\}.$$

At full labor the objective is $-\infty$. For $k > 0$, the cover gives $kg_p \leq 1$, so these controls yield positive consumption and a successor inside the cover. Choose a common bound $\bar{\mathcal{D}} \geq \mathcal{D}_p$ on the primitive neighborhood. If $0 \leq \varphi \leq \bar{\mathcal{D}}$, the stage bound gives $\mathcal{T}_p\varphi \leq (1-\beta)\mathcal{D}_p + \beta\bar{\mathcal{D}} \leq \bar{\mathcal{D}}$. The action $L = 0, z = \beta$ has zero normalized current reward and gives a nonnegative incumbent. Thus every operator preserves the same function interval $[0, \bar{\mathcal{D}}]$, including all iterates starting from zero. On the compact primitive neighborhood, η , $1-\eta$, β and $1-\beta$ have positive lower bounds. Consequently the divergence of $D_\beta(z)$ at either endpoint excludes z near zero or one uniformly from every maximizer; the bounded continuation term cannot offset it. The same argument with $\log(1-L)$ excludes labor near one. Each maximization therefore uses a fixed compact control rectangle. For continuous continuation functions, the objective is continuous in states and controls. Starting from zero, the iterates are jointly continuous in states and primitives by induction.

The operators satisfy $\|\mathcal{T}_p\varphi - \mathcal{T}_p\psi\|_\infty \leq \beta_{\max}\|\varphi - \psi\|_\infty$, with $\beta_{\max} < 1$. Their iterates converge uniformly in states and primitives. At $k = 0$, the resource gain vanishes and cake attains the maximum zero, so every iterate is zero there. At positive stocks, Bellman's equation and normalization make $V_p(\bar{K}k) - E_p(\bar{K}k)$ a bounded fixed point. Uniqueness identifies it with the limit, proving joint continuity at positive stocks. The uniform first action bounds and Bellman's equation then give a jointly closed optimal successor graph and joint upper hemicontinuity.

Choose $x < S_p < y$ arbitrarily close to the frontier, inside its ordered interval. The strict directions $\max G_p(x) < x$ and $\min G_p(y) > y$ persist under small primitive changes. Continuity of the physical root and verification stocks keeps x, y inside the nearby ordered intervals. These directions place every nearby frontier between x and y . Their arbitrary proximity to the original frontier proves continuity. $\square$

B Frontier geometry and allocation

Lemma 7 (Separation of active capital slopes). *At a classified frontier, the extinct and productive optimal history sets are compact and their first successors are strictly separated. Every productive active capital slope exceeds every extinct one.*

Proof. Optimal histories form a compact set by Lemma 1. Classification excludes the current stock from its optimal successors and splits the histories into two closed sets, with successors strictly

below and above the frontier. For Theorem 1, Lemma 5 gives unique global labor maximization and (A.14) gives $r_{p,Ks} > 0$ near S . For Theorem 8, its proof establishes the same properties on $0 < K \leq 3/5$, while $S < \lambda/400 \leq 1/400$. Thus in either case $a_\pi = r_{p,K}(S, s)$ is strictly increasing in s . Compactness and successor separation then give a strictly positive gap between the two sets of slopes. $\square$

Lemma 8 (Capital envelopes at the frontier). *Fix a classified primitive vector p and its frontier S . There are two local restricted Bellman values D^- and D^+ , represented on fixed compact sets of first actions, such that $V_p = \max\{D^-, D^+\}$ near S . Their active actions at S are precisely the first actions of the extinct and productive optimum sets $\mathcal{O}_-$ and $\mathcal{O}_+$, respectively. They satisfy*

$$D^\sigma(S + z) = V_p(S) + \max_{\pi \in \mathcal{O}_\sigma} a_\pi z + o(|z|), \quad \sigma \in \{-, +\}, \quad (\text{B.1})$$

where $a_\pi = \eta\{q + F_{p,K}(S, L_0)\}/C_0$. Consequently

$$V_p(S + z) = V_p(S) + \max_{\pi \in \mathcal{O}_- \cup \mathcal{O}_+} a_\pi z + o(|z|). \quad (\text{B.2})$$

The result allows multiple optimal histories and nondifferentiable optimal policies.

Proof. Write

$$\begin{aligned} c(K, L, s) &= qK + F_p(K, L) - s, \\ \mathcal{Q}_p(K, L, s) &= \eta \log c(K, L, s) + (1 - \eta) \log(1 - L) + \beta V_p(s). \end{aligned}$$

Initial capital affects first consumption; continuation value depends on s alone.

Fixed compact action sets. First represent each branch on a compact action set fixed as capital varies. Lemma 7 and compactness give $d > 0$ such that every extinct active successor at S is at most $S - 2d$, and every productive one is at least $S + 2d$. Restrict the two Bellman problems to $s \leq S - d$ and $s \geq S + d$. Denote their values by $D^-(K)$ and $D^+(K)$. Both equal $V_p(S)$ at S .

Choose a common invariant resource cover $[0, \bar{K}]$ for initial stocks near S , as in Lemma 1. Give $\mathcal{Q}_p$ value $-\infty$ at zero consumption, zero successor or full labor. On the closed feasible first action sets this extension is upper semicontinuous: V_p is continuous at positive stocks, and $V_p(s) \leq E_p(s) + \mathcal{D}_p \rightarrow -\infty$ as $s \downarrow 0$. The restricted maxima are attained. An active action from each branch remains feasible when K is close to S and supplies a common finite lower bound on both restricted values.

Thus all nearby restricted maximizers satisfy $c(K, L, s) \geq 4c_0$, $s \geq 2s_0$, and $L \leq 1 - \varepsilon$ for some $c_0, s_0, \varepsilon > 0$. Current reward and continuation value have common upper bounds. If consumption or leisure vanished, current utility would tend to $-\infty$; if the successor vanished, the cake bound would force continuation value to $-\infty$. Either contradicts the common finite lower bound.

Shrink the stock neighborhood so that $|c(K, L, s) - c(S, L, s)| \leq c_0$ uniformly for $0 \leq L \leq 1$. For each branch use the fixed compact index set

$$\begin{aligned} \mathcal{Q}_\sigma &= \{(L, s) : 0 \leq L \leq 1 - \varepsilon, s_0 \leq s \leq \bar{K}, \\ &\quad c(S, L, s) \geq 2c_0, s \text{ satisfies the branch cut}\}. \end{aligned}$$

Every nearby restricted maximizer lies in its corresponding set, and every indexed action is feasible nearby with consumption at least c_0 . Hence $D^\sigma(K) = \max_{\mathcal{Q}_\sigma} \mathcal{Q}_p(K, L, s)$ exactly. Closedness of the unrestricted optimal successor graph excludes successors in $[S - d, S + d]$ for initial stocks sufficiently close to S . Bellman's equation then gives $V_p = \max\{D^-, D^+\}$. At S , each active action and optimal continuation form an optimal history in the corresponding classified branch.

Uniform expansion and active limits. On these compact sets, consumption is bounded away from zero. The CES production function has continuous first and second capital derivatives for $K > 0$, including its zero labor extension. Uniform Taylor expansion therefore gives

$$\mathcal{Q}_p(S + z, L, s) = \mathcal{Q}_p(S, L, s) + a(L, s)z + O(z^2), \quad a(L, s) = \frac{\eta\{q + F_{p,K}(S, L)\}}{c(S, L, s)},$$

with a common remainder bound. Only first consumption is differentiated; the finite, continuous continuation value is unchanged.

The active set on Q_σ is compact. As $z \rightarrow 0$, uniform convergence makes every limit of nearby maximizers active at S ; continuity of a makes their slopes approach the active slope set. At a nearby maximizer, the objective evaluated at S is at most $V_p(S)$, so the uniform expansion supplies an upper bound. An active action maximizing az supplies a lower bound. Since nearby slopes approach the active slope set, these bounds differ by $o(|z|)$. This gives (B.1); maximizing over the two branches gives (B.2). $\square$

Proof of Theorem 2. Lemma 7 gives successor and slope separation. Lemma 8 gives (B.2) without assuming differentiability of value. The right derivative is therefore the largest active slope and the left derivative the smallest, proving (3.2).

Strong order puts successors chosen below S at or below every successor chosen at S , and those chosen above at or above every such successor. Every limiting successor belongs to the optimal successor set at the frontier by closedness. Order therefore forces the respective limits to be $\min G_p(S)$ and $\max G_p(S)$. The labor equation $s - qK = \zeta(L)$ has $\zeta'(L) = T_L > 0$ on its maximizing interior branch. Since the two limiting successors are distinct, continuity and $\partial L / \partial s = 1/T_L > 0$ give strictly ordered limiting labor.

If an optimal history reached S after its initial date, keep its preceding labor fixed and vary the successor slightly in both directions. Positive consumption makes both variations feasible. The successor objective is a smooth current payoff plus βV_p ; its right derivative exceeds its left derivative by $\beta J_p > 0$. It cannot have a local maximum there. Finally, the one sided expansions give (3.3). $\square$

C Optimal allocations near extinction

Proof of Theorem 4. Fix the primitives and suppress their subscripts on production and value. Write

$$r_E = \theta - 1 \in (0, 1), \quad d = -\rho > 0, \quad c_w = (1 - w)/w, \\ a_p = \eta/(1 - \beta), \quad W_E(K) = V_p(K) - E_p(K).$$

Lemma 1 supplies attained optimal actions, positive consumption and successors, resource exhaustion, and optimal tails. Proposition 3 gives $0 \leq W_E \leq \mathcal{D}_p < \infty$ on all positive stocks. These foundations hold without the classification hypotheses.

A small invariant domain and uniform action bounds. The CES formula gives $F(K, L) \leq m_K K^\theta$ for every labor choice. Put

$$\varepsilon_E = \left(\frac{1 - q}{2m_K} \right)^{1/r_E}, \quad q_E = (1 + q)/2 < 1.$$

Every feasible successor from $0 < K \leq \varepsilon_E$ satisfies $K' \leq q_E K$. Thus every feasible history starting in this interval contracts and remains there.

For $Q(K, L) = qK + F(K, L)$ and $z = K'/Q(K, L)$, define

$$H_E(K, L) = a_p \log \frac{Q(K, L)}{qK} + (1 - \eta) \log(1 - L),$$

Using D_β from (A.2), resource exhaustion and Bellman's equation give

$$W_E(K) = \max_{0 \leq L < 1, 0 < z < 1} \{H_E(K, L) - a_p D_\beta(z) + \beta W_E(zQ(K, L))\}. \quad (\text{C.1})$$

Resource exhaustion gives $C = (1 - z)Q(K, L)$. The omitted control endpoints have value $-\infty$ and cannot maximize.

Since $\log(1 + x) \leq x$ and leisure utility is nonpositive, $H_E(K, L) \leq (a_p m_K / q) K^{r_E}$. Iterate (C.1) along an optimal history starting in the invariant interval. Its discounted terminal remainder is bounded by $\beta^n \mathcal{D}_p$ and tends to zero. Therefore

$$0 \leq W_E(K) \leq C_W K^{r_E}, \quad C_W = \frac{a_p m_K}{q(1 - \beta q_E^{r_E})}. \quad (\text{C.2})$$

At every optimal action in this interval, $W_E(K) \geq 0$, $-\log(1 - L) \geq L$, and the same Bellman equality imply

$$(1 - \eta)L + a_p D_\beta(z) \leq \left(\frac{a_p m_K}{q} + \beta C_W q_E^{r_E} \right) K^{r_E} = C_W K^{r_E}. \quad (\text{C.3})$$

Integrating D'_β from β and using $z(1 - z) \leq 1/4$ gives $D_\beta(z) \geq 2(z - \beta)^2$. Consequently

$$L = O(K^{r_E}), \quad z = \beta + O(K^{r_E/2}), \quad \frac{K'}{K} = g_c + O(K^{r_E/2}). \quad (\text{C.4})$$

All constants are independent of the selected optimal action. In particular optimal labor tends to zero, although the feasible labor domain remains $[0, 1]$.

The leading value gain. We identify the coefficient by bounding optimal welfare above and a feasible plan below. Let $\bar{w}$ and $\underline{w}$ be the upper and lower limits of $W_E(K)/K^{r_E}$ as $K \downarrow 0$. Both are finite and nonnegative by (C.2). Dropping leisure and saving losses from the optimal Bellman equality and using $K'/K \rightarrow g_c$ gives

$$\bar{w} \leq a_p m_K / q + \beta g_c^{r_E} \bar{w}.$$

For the reverse comparison fix $r_E < \gamma_0 < 1$, choose $L = K^{\gamma_0}$ and $z = \beta$, and append an optimal continuation. For sufficiently small K , this action has positive consumption and leisure. Moreover

$$\frac{L}{K} \rightarrow \infty, \quad \frac{F(K, L)}{K^\theta} \rightarrow m_K, \quad \frac{L}{K^{r_E}} \rightarrow 0, \quad \frac{zQ(K, L)}{K} \rightarrow g_c.$$

Its Bellman payoff therefore gives

$$\underline{w} \geq a_p m_K / q + \beta g_c^{r_E} \underline{w}.$$

Since $\beta g_c^{r_E} < 1$, these two inequalities force $\bar{w} = \underline{w} = a_E$. This proves the value expansion in (5.3).

Output, labor and consumption. The value expansion forces each nonnegative leading loss to vanish. Return to equality in (C.1) at an arbitrary optimal action. The resource gain has a uniform

logarithmic remainder $O(K^{2r_E})$, and the leisure logarithm has remainder $O(L^2) = O(K^{2r_E})$. Using the value expansion and (C.4) therefore gives

$$\frac{a_p}{q} \left(m_K - \frac{F(K, L)}{K^\theta} \right) + (1 - \eta) \frac{L}{K^{r_E}} + a_p \frac{D_\beta(z)}{K^{r_E}} \longrightarrow 0. \quad (\text{C.5})$$

Each term is nonnegative. The limit is uniform over optimal actions: the bounds and expansions used above hold for every such action, and every successor tends to zero with its initial stock. It follows that $F(K, L)/K^\theta \rightarrow m_K$. For positive labor the exact production formula is

$$F(K, L) = m_K K^\theta [1 + c_w(K/L)^d]^{-\theta/d}.$$

Thus every sufficiently small optimal action has $L > 0$ and $L/K \rightarrow \infty$. Equation (C.3) also gives $L \rightarrow 0$. Resource exhaustion and $z \rightarrow \beta$ now imply $C/K \rightarrow (1 - \beta)q$.

At these actions labor is strictly between zero and one. Varying it while keeping the optimal successor fixed is feasible locally because consumption is positive. The resulting necessary condition is

$$F_L(K, L) = \nu \frac{C}{1 - L}.$$

Only current utility is differentiated. From the exact CES derivative,

$$F_L(K, L) = \theta m_K c_w K^{\theta+d} L^{-1-d} [1 + c_w(K/L)^d]^{-1-\theta/d}.$$

The limits already proved imply

$$L^{1+d} \sim \frac{\theta m_K c_w}{q\nu(1 - \beta)} K^{\theta+d-1}.$$

Taking positive roots proves $L \sim \ell_E K^{\gamma_E}$ uniformly. The identities $\gamma_E - r_E = d(1 - r_E)/(1 + d) > 0$ and $1 - \gamma_E = (1 - r_E)/(1 + d) > 0$ give the stated exponent ordering. This completes the optimal action limits.

The constant on an extinct tail. Every extinct optimum enters $(0, \varepsilon_E]$ at a finite date t_0 and obeys $K_{t_0+j} \leq q_E^j K_{t_0}$ thereafter. Equation (C.4) gives

$$\frac{K_{t+1}}{g_c K_t} = 1 + e_t, \quad |e_t| \leq C K_t^{r_E/2} \quad (t \geq t_0)$$

for a fixed finite C . The geometric stock bound makes $\sum_{t \geq t_0} |e_t|$ finite. Every factor $1 + e_t$ is positive; eventually $|e_t| < 1/2$, so the sum of their logarithms converges to a finite real number. Hence their product has a strictly positive finite limit. The identity

$$\frac{K_t}{g_c^t} = \frac{K_{t_0}}{g_c^{t_0}} \prod_{j=t_0}^{t-1} (1 + e_j)$$

defines $\Xi_\pi \in (0, \infty)$. The consumption, labor and output limits in (5.4) follow by substituting this capital limit into (5.3). $\square$

D Productivity scaling

The comparison uses a bound on every history that remains below a given stock. It follows from a supporting line to consumption utility.

Lemma 9 (An affine bound on staying welfare). *Let $\eta > 0$, $k > 0$ and $\beta G \geq 1$. Suppose every feasible action from $0 < x \leq k$ satisfies $C + y \leq Gx$ and earns $\eta \log C + \psi(L)$, with $\psi(L) \leq 0$. Set*

$$c_* = (G - 1)k, \quad B_k(x) = \frac{\eta}{1 - \beta} \log c_* + \frac{\eta G}{c_*} (x - k). \quad (\text{D.1})$$

Then B_k is a bounded regional Bellman barrier on $(0, k]$. Every history remaining in that interval has welfare at most B_k at its initial state.

Proof. Since $\beta G \geq 1$ and $\beta < 1$, one has $G > 1$ and $c_* > 0$. For a staying action with finite reward, the supporting line to $\log C$ at c_* gives

$$\begin{aligned} \eta \log C + \psi(L) + \beta B_k(y) - B_k(x) &\leq \frac{\eta}{c_*} (C - c_*) + \frac{\beta \eta G}{c_*} (y - k) - \frac{\eta G}{c_*} (x - k) \\ &\leq \frac{\eta(\beta G - 1)}{c_*} (y - k) \leq 0. \end{aligned}$$

The second inequality uses $C + y \leq Gx$; the constants cancel because $c_* = (G - 1)k$. An action with reward $-\infty$ satisfies the inequality directly. The affine function is bounded on $(0, k]$, so its discounted terminal term vanishes. Telescoping proves the welfare bound. $\square$

Proof of Theorem 3. Put $d = -\rho$, $r = \theta - 1$, $c_K = w^{-\theta/d}$, $c_w = (1 - w)/w$, and $m = D - \delta = \beta^{-1} - 1 > 0$. For $z = K/L > 0$ and $y = c_w z^d$, the production identity is

$$F_p(K, L) = A c_K K^\theta (1 + y)^{-\theta/d}. \quad (\text{D.2})$$

An auxiliary stationary allocation. For $z > 0$ and $y = c_w z^d$, define

$$\begin{aligned} L(z) &= \frac{Dy}{\nu m + D(\nu + 1)y}, \quad K(z) = zL(z), \quad C(z) = (m + Dy)K(z), \\ A_B(z) &= \frac{D(1 + c_w z^d)^{1+\theta/d}}{c_K K(z)^r}. \end{aligned} \quad (\text{D.3})$$

At productivity $A = A_B(z)$, (D.2) gives $F_p(K(z), L(z))/K(z) = D(1 + y)$. Since $m = D - \delta$, $C(z) = F_p(K(z), L(z)) - \delta K(z) > 0$. Thus these formulas construct a feasible stationary allocation with $0 < L(z) < 1$. We use its stock to bound welfare from staying below it; a feasible expansion will beat that bound. Write $a_m = D(\nu + 1)/(\nu m)$ and $b_m = 1 + d(2 - \theta) - r a_m$. Direct differentiation yields

$$\frac{d \log A_B}{d \log z} = \frac{a_m(d + 1)y^2 + b_m y - r(d + 1)}{(1 + y)(1 + a_m y)}.$$

The numerator has positive leading coefficient and negative constant term, hence exactly one positive root y_c . Thus A_B decreases strictly on $0 < y < y_c$ and tends to infinity as $z \downarrow 0$. For all sufficiently large A , the equation $A_B(z) = A$ therefore has a unique solution with $0 < y < y_c$. Along this branch $z, y, L(z), K(z) \rightarrow 0$.

Write $(k, l, c) = (K(z), L(z), C(z))$ at the chosen productivity.

The staying bound. For sufficiently large A , the branch stock satisfies $k < \min\{b, k_a\}$, where b is the classification anchor and k_a is defined in (2.4). Average product at full labor increases below k_a . Thus every feasible planner action from $0 < x \leq k$ satisfies $C + s \leq Gx$, where $G = q + F_p(k, 1)/k$. Since $l < 1$,

$$G > q + F_p(k, l)/k = q + D(1 + y) > \beta^{-1}. \quad (\text{D.4})$$

Lemma 9 applies with $\psi(L) = (1 - \eta) \log(1 - L)$ and $c_* = (G - 1)k$. It bounds every history remaining in $(0, k]$, including every extinct history that decreases from k . A zero successor forces zero consumption thereafter and hence lifetime welfare $-\infty$, below any finite feasible plan.

A feasible expansion beats the bound. For a fixed $0 < v < m$, use labor l to raise capital from k to $(1 + v)k$, then keep that stock and labor forever. Dividing first and subsequent consumption by k gives

$$\begin{aligned} P_v(y) &= m + Dy - v, \\ Q_v(y) &= \frac{D(1 + y)^{1+\theta/d}(1 + v)^\theta}{[1 + (1 + v)^d y]^{\theta/d}} - \delta(1 + v). \end{aligned} \quad (\text{D.5})$$

Write $H_v = D(1 + v)^\theta - \delta(1 + v)$ and $M_c = G - 1$. Along the branch,

$$M_c = \frac{D(1 + y)^{1+\theta/d}}{[1 + c_w k^d]^{\theta/d}} - \delta \longrightarrow m, \quad Q_v(y) \longrightarrow H_v.$$

At $v = 0$, $H_0 = m > 0$. Where the plan is feasible its payoff $W(k)$ has the exact surplus

$$\frac{1 - \beta}{\eta} [W(k) - B_k(k)] = (1 - \beta) \log \frac{P_v(y)}{M_c} + \beta \log \frac{Q_v(y)}{M_c} + \nu \log(1 - l). \quad (\text{D.6})$$

As $y \downarrow 0$, this expression tends to

$$\Psi(v) = (1 - \beta) \log \frac{m - v}{m} + \beta \log \frac{H_v}{m}.$$

It satisfies $\Psi(0) = 0$ and

$$\Psi'(0) = \frac{\beta(\theta - 1)D}{m} > 0.$$

Choose a fixed sufficiently small $v \in (0, m)$ with $H_v > 0$ and $\Psi(v) > 0$. Continuity gives $0 < y_* < y_c$ such that, whenever $0 < y < y_*$, both consumptions in (D.5) and the surplus in (D.6) are positive, and $k < \min\{b, k_a\}$. With $z_* = (y_*/c_w)^{1/d}$, set

$$A_M = \max\{A_*, A_B(z_*)\}. \quad (\text{D.7})$$

This finite threshold ensures all the comparisons for $A > A_M$. The choice of y_* follows from the limiting argument; no smaller classification threshold is obtained.

Frontier placement and productivity order. Capacity output at k exceeds depreciation. Since $k < k_a$, its lower replacement root satisfies $\kappa_A < k$. If $k \leq S_A$, Theorem 1 supplies an attained extinct optimum from k , including at equality. Its capital path strictly decreases, so its welfare is at most $B_k(k)$. The feasible expansion earns more, a contradiction. Therefore $S_A < k < b$. Classification gives initial accumulation and eventual permanence in $[h, \bar{K}(A)]$.

At the physical root and auxiliary stock, respectively,

$$\delta = F_p(\kappa_A, 1)/\kappa_A < Ac_K \kappa_A^r, \quad Ac_K k^r = D(1+y)^{1+\theta/d} < D(1+y_*)^{1+\theta/d}.$$

Set $c_+ = [D(1+y_*)^{1+\theta/d}/c_K]^{1/r}$. These inequalities, together with $\kappa_A < S_A < k$, prove (4.3) and (4.4). As $A \rightarrow \infty$, $y \rightarrow 0$ and $k \rightarrow 0$. Since $\kappa_A < k$, the physical root also tends to zero. Its exact identity

$$Ac_K \kappa_A^r = \delta[1 + c_w \kappa_A^d]^{\theta/d}$$

gives the limit of $A^{1/r} \kappa_A$ in (4.5). The auxiliary identity also gives

$$A^{1/r} k = \left[\frac{D(1+y)^{1+\theta/d}}{c_K} \right]^{1/r} \rightarrow (D/c_K)^{1/r}.$$

Taking lower and upper limits in $\kappa_A < S_A < k$ proves (4.6). Dividing by κ_A and using its positive scaled limit proves (4.7). $\square$

E Barrier construction, completeness and margins

E.1 Calibrated finite block construction

Proof of Theorem 6. The monotone operator $\mathcal{F}_U$ preserves $[0, c_N]$. Since $C_N \leq c_N$, all iterates in (6.10) lie there. The cap can be read as an auxiliary stopping option: stop at cost c_N or continue along a staying edge. Unfolding $\mathcal{F}_U^N g_0$ gives the infimum over early stops and full blocks. A path stopped after $n \in \{0, \dots, N-1\}$ edges has cost $\sum_{t < n} \beta^t d_U(e_t) + \beta^n c_N$; a full N edge prefix has cost $\sum_{t < N} \beta^t d_U(e_t) + \beta^N C_N$. Nonnegative residuals make every early stop cost at least

$$\beta^n c_N = C_N \beta^{n+1-N} \geq C_N.$$

A full prefix costs at least $\Delta_{\text{blk}} + \beta^N C_N = C_N$. If continuation is impossible, the stopping option remains available. Infinite loss edges cannot reduce an infimum, and no attainment is required. Thus $g_N \geq g_0$. Apply monotonicity to each iterate separately, without interchanging $\mathcal{F}_U$ and a maximum, to obtain

$$\mathcal{F}_U \ell_U \geq \max_{1 \leq j \leq N} g_j \geq \max_{0 \leq j < N} g_j = \ell_U.$$

In particular $C_N \leq \ell_U \leq c_N$, and on every staying edge $\ell_U(x) \leq d_U(e) + \beta \ell_U(y)$. For an edge with finite reward, rearrange this inequality using the definition of d_U to obtain

$$B(x) = U(x) - \ell_U(x) \geq r(e) + \beta[U(y) - \ell_U(y)].$$

An edge with reward $-\infty$ satisfies the barrier inequality directly. Both U and ℓ_U are bounded, so the discounted terminal value of B tends to zero along every staying history. The witness calibration gives $W_M - B \geq \ell_U - \alpha \geq C_N - \alpha$. This staying margin and the independently established adverse comparisons give the full certificate. Theorem 5 then establishes selection.

For sharpness, fix any $0 \leq \alpha < C_N$ and put $\Delta = \Delta_{\text{blk}} - (1 - \beta^N)\alpha > 0$. Take N staging states $x_0, \dots, x_{N-1}$ on a cycle. The first $N-1$ staying edges earn zero; the last earns $-\Delta/\beta^{N-1}$. Each state also has free entry into a zero reward target absorber. Nonpositive rewards bound all welfare

by zero, and the entry witnesses attain that bound. Thus $V = W_M = W_H = 0$. Use the regional reference $U \equiv \alpha$. A block starting at x_i has reference loss

$$\Delta\beta^{-i} + (1 - \beta^N)\alpha \geq \Delta_{\text{blk}}, \quad 0 \leq i < N.$$

The eternal cycle from x_0 has welfare

$$-\frac{\Delta}{1 - \beta^N} = -(C_N - \alpha).$$

Every valid regional barrier dominates this stayer by its terminal condition, while every feasible witness floor is at most zero. Consequently no uniform staying margin can exceed $C_N - \alpha$, and the construction attains that bound. For $N > 1$ and $\alpha = 0$, this is an example with zero minimum single period residual and positive block loss. Finally add an edge from the target, earning $-\varepsilon_{\text{ad}}$, to an outside zero reward absorber. Its maximum adverse comparison slack is ε_{ad} : the target witness cannot exceed zero and the continuation ceiling cannot be below the outside value zero. The zero ceiling attains that slack. This proves sharpness of the full margin $\min\{C_N - \alpha, \varepsilon_{\text{ad}}\}$ for every admissible α . $\square$

E.2 Compact completeness and sharp margins

Maintain the assumptions and exact loss profile of Subsection 6.4. We first connect finite block losses to the least loss from perpetual staying, then obtain the converse and margin suprema. Prefix classes contain every staying edge, including edges of infinite loss and prefixes that cannot continue indefinitely.

Lemma 10 (Compact loss profile). *One has*

$$\delta_T \uparrow G, \quad G = \sup_{T \geq 1} \frac{\delta_T}{1 - \beta^T}. \quad (\text{E.1})$$

If $G < \infty$, an eternal staying path attains it. An infinite zero loss staying path exists if and only if $\delta_T = 0$ for every finite T .

Proof. Matching successive endpoints is a closed condition. Each finite prefix class is therefore compact, and its lower semicontinuous loss attains a minimum whenever the class is nonempty. Nonnegative losses make δ_T nondecreasing. Put $g = \lim_T \delta_T$. Every eternal staying path has total loss at least g .

If $g < \infty$, the cylinders of compatible T edge prefixes with loss at most g are nonempty, closed and nested in $\mathcal{X}^{\mathbb{N}}$. Compactness gives a path in their intersection, whose total loss is at most g . Thus $G = g$, with attainment. If $g = +\infty$, every eternal staying path has infinite loss or none exists, so again $G = g$. Splitting an eternal path into blocks of T edges gives loss at least $\delta_T/(1 - \beta^T)$. This bound and $\delta_T \uparrow G$ prove the supremum formula, including empty path classes. Finally, the attained minimum is zero exactly when every finite prefix minimum is zero. $\square$

The compact extraction follows the finite horizon viability argument of Bertsekas (1972, Proposition 4). If every finite horizon admitted zero staying loss, an infinite zero loss path would exist. The converse below uses this implication to obtain a positive finite block loss.

Proof of Theorem 7, part 1. Suppose every optimizer is core. A zero loss adverse edge has finite reward and finite successor value. Appending an attained optimal continuation would give a

noncore optimum, a contradiction. Thus every adverse loss is positive. An eternal zero loss staying path would also attain value: telescoping its Bellman losses and using bounded $V|_M$ gives its welfare as V at its initial state. Lemma 10 therefore gives $G > 0$.

Fix a finite $0 < \gamma < G$. The compact loss profile supplies N and a finite $0 < \Delta_{\text{blk}} \leq \delta_N$ with $C_N = \Delta_{\text{blk}}/(1 - \beta^N) > \gamma$. Apply Theorem 6 with $U = V|_M$ and zero calibration error. Its bounded regional barrier has staying margin at least C_N . Because every optimizer is core and value is attained, the core witness floors can be chosen as $W_M = V|_M$ and $W_H = V|_H$. The exact ceiling $\bar{V} = V$ gives adverse slack $d(e) > 0$, completing a pointwise strict certificate.

If the adverse graph is nonempty and compact and its loss is lower semicontinuous, its positive losses have a positive minimum. An infinite minimum, or an empty adverse graph, permits any finite positive bound. Combining such a bound with C_N gives a uniform strict certificate. Theorem 5 proves the reverse implications from pointwise or uniform strict certificates to selection. $\square$

Proof of Theorem 7, part 2. Under selection, the preceding construction supplies a bounded barrier for every finite staying margin $0 < \gamma < G$. Margin zero uses $B = V|_M$, which satisfies the staying Bellman inequalities and has vanishing discounted terminal value.

If $G < \infty$, let an eternal staying path attain its loss G . Bounded $V|_M$ and telescoping give this path finite welfare $V(x) - G$ at its initial state x . Every valid regional barrier, bounded or not, dominates that welfare by its terminal condition. Consequently $V(x) - B(x) \leq G$, and no uniform staying margin with exact floors can exceed G . Together with the construction for every smaller finite margin, this proves $\Gamma_M = G$. When $G = +\infty$, the construction gives arbitrarily large finite staying margins and yields the same equality.

For a full certificate, every feasible witness floor satisfies $W_i(x) \leq V(x)$, and every legitimate continuation ceiling satisfies $\bar{V}(y) \geq V(y)$. When $G < \infty$, the minimizing path's initial state therefore rules out a uniform staying margin greater than G for any full certificate. On every adverse edge of finite loss,

$$W_i(x) - \{r(e) + \beta\bar{V}(y)\} \leq V(x) - \{r(e) + \beta V(y)\} = d(e).$$

Edges of infinite loss impose no finite upper bound. Taking the infimum gives $\Gamma \leq \min\{G, a_*\}$ without assuming that the adverse infimum is attained.

Conversely, exact core witnesses and exact adverse ceilings attain each adverse slack $d(e)$. A bounded barrier with staying margin greater than any prescribed finite $0 < \varepsilon < \min\{G, a_*\}$ therefore gives a full certificate of margin ε . There is always a weak certificate of margin zero using $B = V|_M$. These observations prove $\Gamma = \min\{G, a_*\}$, including $a_* = 0$, empty adverse graphs and infinite suprema. The formula specifies a supremum, without asserting endpoint attainment. $\square$

F The harmonic CES application

F.1 The frontier bound

Proof of Theorem 8. All terminating decimals below denote exact rational numbers. The finite comparisons use the 32-term logarithm bounds and square-root enclosures with denominator 10^{24} in Lemma 11. Write $q_- = .919, q_+ = .921, \beta_- = .969, \beta_+ = .971$, and $\eta_- = .699, \eta_+ = .701$. Set $D_+ = 1/.969 - .919$. The foundations in Lemma 1 apply to this family without the productivity restriction $A > A_*$. We establish labor choice and successor order, exclude low fixed optimal transitions, and construct productive capture. These give classification; a sharper feasible witness then locates the frontier.

Full labor choice and local order. For fixed capital K and successor s , let $r(K, s)$ be the maximum current payoff over the full feasible labor domain. Put $\alpha = \eta/(1 - \eta) > 1$. Whenever zero labor is feasible, choosing $L = K$ instead increases payoff by at least

$$h(K) = \eta \log(1 + d\sqrt{K}) + (1 - \eta) \log(1 - K), \quad d = A/q > 4/5, \quad 0 < K \leq 3/5.$$

As a function of $x = \sqrt{K}$, its derivative has the sign of $\eta d - 2(1 - \eta)x - d(2 - \eta)x^2$, which decreases strictly. Thus h has no interior minimum. Its limit at zero is zero, and

$$h(3/5) > \frac{69}{100} \frac{2t}{2+t} - \frac{31}{100} \left\{ \sum_{j=1}^4 \frac{(3/5)^j}{j} + \frac{(3/5)^5}{5(2/5)} \right\} = \frac{81903873}{2046250000} > 0, \quad t = \frac{4}{5} \frac{774}{1000}.$$

Here $\sqrt{3/5} > .774$, $\log(1+t) \geq 2t/(2+t)$, and the displayed finite sum bounds $-\log(1 - 3/5)$ above. Zero labor is therefore dominated throughout this stock interval.

Let $F = F_A(K, L)$ and $\zeta = F - \alpha(1 - L)F_L$. The labor derivative of payoff has the sign of $s - qK - \zeta$, and

$$\frac{\zeta_L}{F} = \frac{3K\mathcal{D}_\alpha}{4L^2(K+L)^2}, \quad \mathcal{D}_\alpha = (2 + 3\alpha)KL - \alpha K + 2(1 - \alpha)L^2 + 4\alpha L.$$

The derivative $(\mathcal{D}_\alpha)_L = (2 + 3\alpha)K + 4\{\alpha(1 - L) + L\}$ is positive. Hence ζ decreases and then increases. At most one interior critical point can be a local maximum of payoff. Zero labor is dominated, while zero consumption and full labor give payoff $-\infty$. The global labor maximizer is therefore unique, interior, and satisfies $\zeta_L > 0$.

On this branch, differentiating the first order condition gives

$$\text{sgn } r_{Ks} = \text{sgn} \left\{ q\mathcal{Q} + \frac{3FL(2 - L)}{K} \right\}, \quad \mathcal{Q} = K(3L - 1) + 2L(2 - L).$$

In fact $\mathcal{Q} \geq 0$ at every positive global labor maximizer. Otherwise $L < 1/3$ and $K(1 - 3L) > 2L$, so the labor elasticity $\epsilon_L = LF_L/F = 3K/[2(K + L)]$ satisfies $\epsilon_L(1 - L) > 1$. The first order condition gives $F/C = L/[\alpha(1 - L)\epsilon_L] < L/\alpha < 1$. Zero labor at the same successor is then feasible, with consumption $C - F$, and the gain from positive labor, divided by η , is

$$\log \frac{C}{C - F} + \alpha^{-1} \log(1 - L) < -\log(1 - L/\alpha) + \alpha^{-1} \log(1 - L) \leq 0.$$

The last inequality is concavity of logarithm, contradicting global maximization. Consequently $r_{Ks} > 0$ for $0 < K \leq 3/5$. Integration on feasible crossed rectangles and cancellation of continuation values in the Bellman comparison give

$$\max G(x) \leq \min G(y) \quad (0 < x < y \leq 3/5). \quad (\text{F.1})$$

These arguments include every feasible labor choice.

No fixed optimal transition at low stocks. Patience holds because $\beta_-[3/2 - q_+/2] > 1$. Use $D = \beta^{-1} - q$, $E = D - \frac{3}{2}(1 - q) < 0$, and $\nu = (1 - \eta)/\eta$ as in (2.2). If $K \in G(K)$, repeat its

maximizing action. A two date saving variation and the interior labor condition imply $F_K = D$ and, with $z = K/L$,

$$z > -E/D, \quad L = \frac{(3/2)Dz}{\nu E + (3/2 + \nu)Dz}, \quad A^2 = \frac{D^2(1+z)^5}{18Lz}.$$

Put

$$z_- = \frac{14551}{215102}, \quad \nu_+ = \frac{301}{699}, \quad L_- = \frac{3/2}{3/2 + \nu_+}, \quad z_+ = \frac{3/5}{L_-}.$$

The parameter bounds give $z > z_-$, $L > L_-$, and $D \leq D_+$. If $K \leq 3/5$, then $z < z_+$, and direct rational comparison gives

$$A^2 < \frac{D_+^2(1+z_+)^5}{18L_-z_-} < \frac{1}{4}.$$

This contradicts $A \geq .749$. Thus $K \notin G(K)$ on $(0, 3/5]$.

Scaling and a productive witness. For $\lambda = (A_0/A)^2 \leq 1$,

$$F_A(\lambda K, L) = \lambda A_0 \left(\frac{2KL}{\lambda K + L} \right)^{3/2} \geq \lambda F_{A_0}(K, L). \quad (\text{F.2})$$

Scale every stock of a feasible baseline history by λ , keep labor unchanged, and consume residual resources. Each consumption is at least its baseline value times λ . Its welfare is therefore at least baseline welfare plus $\eta \log \lambda / (1 - \beta)$. For any history, normalized consumption $\hat{C} = C/\lambda$ gives the exact decomposition

$$J = \frac{\eta}{1 - \beta} \log \lambda + \sum_{t \geq 0} \beta^t \{ \eta \log \hat{C}_t + (1 - \eta) \log(1 - L_t) \}. \quad (\text{F.3})$$

The shift always uses the actual η, β .

At A_0 , a baseline witness from 1/20 sets

$$k_j = \min\{(13/10)^j/20, 5/2\} \quad (0 \leq j \leq 15),$$

uses labor 3/4 for its first fifteen transitions, then maintains 5/2 at labor .77. Consumption is the resource residual. At A_0, q_- , every initial consumption exceeds .00245, and tail consumption exceeds .75436. The exact logarithm bounds of Lemma 11 give the following uniform stage payoff floors:

$$\begin{aligned} (v_0, \dots, v_{14}) &= (-4.63, -3.98, -3.51, -3.13, -2.80, -2.51, -2.25, -2.03, \\ &\quad -1.84, -1.68, -1.57, -1.51, -1.52, -1.64, -1.72), \quad v_\infty = -.64, \\ &\quad \sum_{j=0}^{14} \beta_+^j v_j + \frac{\beta_+^{15} v_\infty}{1 - \beta_+} > -46. \end{aligned}$$

For each date these floors use lower consumption at A_0, q_- and the smaller payoff at the two endpoints of η . The floors are negative, so discounting them at β_+ preserves a lower bound. The last displayed inequality is rational.

If an optimum from $b_A = \lambda/20$ decreased capital, (F.1) and exclusion of fixed transitions would force its subsequent stocks to decrease. Along that history,

$$\widehat{C}_t \leq .921(.05) + .749(.1)^{3/2} < .07.$$

Since $\log(.07) < -5/2$, its normalized welfare in (F.3) would be below $-(.699)(5/2)/(1 - .969) = -3495/62 < -50$. The scaled witness earns more than the same shift minus 46, a contradiction. Thus every optimal successor from b_A is larger. Order then gives positive drift throughout $[b_A, 3/5]$: a downward optimal history from that interval would remain above b_A , decrease to a positive limit, and create a fixed optimal transition by closedness of G . The same argument rules out perpetual residence in $[b_A, h_A)$.

Retention and classification. Starting from $s + \varepsilon$, keep an optimum's first labor and successor from s , consuming the extra resources. For $\Phi_A(s) = qs + F_A(s, 1)$, this gives

$$V(s + \varepsilon) - V(s) \geq \eta \log\{1 + q\varepsilon/\Phi_A(s)\}.$$

An optimal action from $K \geq .6$ with $0 < s \leq h_A$ could therefore be improved by increasing saving slightly whenever $\beta qC > \Phi_A(s)$. This condition holds uniformly:

$$\beta qC \geq .969(.919)\{.919(.6) - .18\} = .3307357854 > .327564\lambda \geq \Phi_A(s).$$

The final inequality follows from $\Phi_A(h_A) \leq \lambda\{.921(.18) + .749(.36)^{3/2}\}$. A zero successor has welfare $-\infty$. Together with positive drift below $.6$, this proves retention above h_A . Since $F_A(K, L) \leq 2^{3/2}A < 3A$, the resource recursion makes $[0, \overline{K}_A]$ invariant and gives finite entry into it from every larger finite stock.

At a positive replacement root,

$$(1 - q)^2 = \frac{8A^2K}{(1 + K)^3}.$$

Full-capacity replacement is strict at $K = \lambda(.0015)$: by (F.2), it suffices to check

$$\frac{8A_0^2(.0015)}{(1.0015)^3} > (.081)^2.$$

Proposition 1 therefore supplies the lower physical root, with

$$\frac{(.079)^2}{8A^2} < \kappa_A < .0015\lambda < b_A.$$

Every finite welfare history at or below κ_A first moves strictly below it and then declines to zero. Apply Lemma 3 between κ_A and b_A , using (F.1), absence of fixed transitions, and the productive anchor just proved. This gives complete classification, coexistence, and the stated successor directions.

A sharper witness and the frontier bound. Let $k_0 = 1/400$. Every history confined below λk_0 , after normalization, obeys

$$\widehat{C}_t + \widehat{K}_{t+1} \leq R\widehat{K}_t, \quad R = 1.027, \quad .921 + .749 2^{3/2} \sqrt{.0025} < R.$$

All normalized consumptions are below $Rk_0 < 1$. Dropping leisure and replacing η, β by $e = .699, d = .969$ therefore bounds normalized welfare above by that of the linear resource problem. Its value at k_0 is

$$\mathcal{E}_R(k_0) = \frac{e}{1-d} \log((1-d)Rk_0) + \frac{ed}{(1-d)^2} \log(dR) < -216. \quad (\text{F.4})$$

The policy $K' = dRK$, $C = (1-d)RK$ and its Bellman inequality give this formula. The terminal comparison is valid for every finite welfare history under consideration: summability of $\beta^t(-\log \hat{C}_t)$ implies summability with discount $d \leq \beta$, and $\hat{C}_t \leq R\hat{K}_t$ then gives $d^t \log \hat{K}_t \rightarrow 0$. Histories of welfare $-\infty$ cannot beat a finite witness.

The following witness beats (F.4) and has rational stocks at every date. For rational $x \geq 0$, put

$$\begin{aligned} \rho_{24}(x) &= 10^{-24} \lfloor 10^{24} \sqrt{x} \rfloor, \\ R_j^- &= .919k_j + .749 \rho_{24} \left(\left[\frac{(3/2)k_j}{k_j + 3/4} \right]^3 \right), \\ k_{j+1} &= 10^{-12} \lfloor 10^{12} (.99) R_j^- \rfloor \quad (0 \leq j < 44). \end{aligned}$$

Choose labor $3/4$, consume the residual, and at k_{44} append the preceding witness from $1/20$, retaining its first successor .065. The extra initial stock raises its first consumption. The recurrence and the arithmetic bounds in Lemma 11 give

$$\begin{aligned} k_{j+1} - k_j &> .00003538, \quad R_j^- - k_{j+1} > .0000256 \quad (j < 44), \\ k_{43} &= .047313081699 < .05 < k_{44} = .062737579046. \end{aligned}$$

For an explicit payoff certificate, let

$$u_j^- = \min_{e \in \{.699, .701\}} \{e \underline{\log}_{32}(R_j^- - k_{j+1}) + (1-e) \underline{\log}_{32}(1/4)\},$$

where $\underline{\log}_{32}$ is the rational lower logarithm bound with 32 terms and binary range reduction from that lemma. Table 2 reports exact recurrence endpoints and rational payoff floors $g_i < \sum_{j=0}^3 \beta_+^j u_{4i+j}^-$, for blocks $i = 0, \dots, 10$. Thus each row requires four substitutions into the stated recurrence and logarithm formula; no optimized policy or value function is used.

Table 2: Exact checkpoints for the 44 transition frontier witness. Capital endpoints are multiplied by 10^{12} .

Dates	Capital endpoint $10^{12}k_{4i+4}$	Payoff floor g_i
0–3	2656429589	−29.91
4–7	2861544059	−29.73
8–11	3136449958	−29.50
12–15	3515325937	−29.22
16–19	4056717910	−28.87
20–23	4868178627	−28.42
24–27	6165754241	−27.84
28–31	8437689848	−27.09
32–35	12981176922	−26.06
36–39	24149629302	−24.61
40–43	62737579046	−22.44

Every floor is negative. The complete baseline witness therefore earns at least

$$\sum_{i=0}^{10} \beta_+^{4i} g_i - 46\beta_+^{44} > -197.813 > -200.$$

Scaling by (F.2) preserves this lower bound apart from the actual-parameter shift in (F.3). A downward optimum from $\lambda/400$ would remain below that stock by (F.1), contradicting the upper bound -216 and the witness floor -200 . Classification therefore gives $S_A < \lambda/400$.

Finally,

$$\frac{S_A}{\kappa_A} < \frac{A_0^2}{50(.079)^2} < 1.798.$$

This rational comparison completes the frontier bound. $\square$

F.2 An analytical continuation bound

The certificate requires an upper bound on every continuation, including those that leave the staging region. The following construction supplies that bound without solving the Bellman equation.

Lemma 11 (Rational bounds for elementary functions). *Let $x > 0$ be rational. If integers $M, j \geq 0$ and $n \geq 1$ satisfy $j^n \leq 10^{nM}x < (j+1)^n$, then $j10^{-M} \leq x^{1/n} < (j+1)10^{-M}$. For $1 \leq x \leq 2$, put $v = (x-1)/(x+1)$. For every integer $N \geq 1$,*

$$2 \sum_{j=0}^{N-1} \frac{v^{2j+1}}{2j+1} \leq \log x \leq 2 \sum_{j=0}^{N-1} \frac{v^{2j+1}}{2j+1} + \frac{2v^{2N+1}}{(2N+1)(1-v^2)}. \quad (\text{F.5})$$

Binary range reduction extends these bounds to every positive rational argument.

Proof. The root bounds follow by taking powers. Integrating the geometric series for $2/(1-v^2)$ gives the logarithm series; replacing every omitted denominator by $2N+1$ bounds its positive tail. Write an arbitrary argument as $2^k x$, with $1 \leq x \leq 2$, and bound $k \log 2 + \log x$, reversing the endpoints when $k < 0$. $\square$

All decimal constants in this appendix denote rational numbers. The scalar bounds below follow from Lemma 11 with $N = 30$ and root brackets of width 10^{-30} . For the partial fractions it suffices to use

$$1.4142135623730950488 < \sqrt{2} < 1.4142135623730950489,$$

whose validity follows by squaring.

Lemma 12 (An unrestricted continuation ceiling). *For the parameter box (7.3), set*

$$f_0 = \frac{1}{20}, \quad \psi(K) = \frac{2}{5}K - \frac{5}{\sqrt{K}} - \frac{37}{2}, \quad \alpha = 7 - 2\sqrt{6}.$$

The function

$$\Psi(K) = \begin{cases} \psi(\max\{K, f_0\}), & 0 \leq K \leq \alpha, \\ .9K - 23, & K > \alpha \end{cases} \quad (\text{F.6})$$

satisfies, for every feasible choice with $K > 0$, $C > 0$ and $0 \leq L < 1$,

$$\eta \log C + (1 - \eta) \log(1 - L) + \beta \Psi(K') \leq \Psi(K) - \frac{1}{1000}. \quad (\text{F.7})$$

Choices with $C = 0$ or $L = 1$ have reward $-\infty$ and also satisfy the inequality. The sharper ceiling

$$\Psi^\sharp(K) = \Psi(K) - \frac{10}{301} \quad (\text{F.8})$$

satisfies $V(K) \leq \Psi^\sharp(K)$ at every finite positive stock.

Proof. The departure comparisons need a tight ceiling at low stocks, but subsequent recovery may visit large stocks. The curved branch supplies the low stock bound; the affine branch covers large stocks. The floor keeps the ceiling finite near zero, and checking transitions across the join preserves unrestricted coverage. A valid terminal limit then bounds whole histories, while the uniform residual slack permits the downward shift.

We verify the curved test up to capital $9/4$ and the affine test globally. A decreasing supporting price reduces the curved calculation to a scalar resource inequality, proved by two sign arguments and an endpoint check. Write

$$A_+ = .751, \quad q_+ = .9201, \quad \beta_0 = .9699, \quad \beta_+ = .9701, \quad d = .2999, \quad a = .4, \quad \ell_0 = \frac{25}{32},$$

and use $F(K, L) = A_+[2KL/(K + L)]^{3/2}$ for upper bounds. Since leisure utility is nonpositive, its smallest coefficient gives an upper bound.

The floor. Let $P(K) = \psi(\max\{K, f_0\})$ and $\Phi_+(K) = q_+K + F(K, 1)$. For $K \geq f_0$ and $K' \leq f_0$, dropping leisure utility bounds the residual relative to P by $\eta \log \Phi_+(K) + \beta_0 \psi(f_0) - \psi(K)$. The elasticity $K\Phi'_+/\Phi_+$ is at most $3/2$, and

$$\min_{K>0} \left(aK + \frac{5}{2\sqrt{K}} \right) = 3a^{1/3}(5/4)^{2/3} > \frac{3}{2}(.7001).$$

Thus this residual decreases in K . At the floor, $\Phi_+(f_0) < .0681$ and $-\psi(f_0) < 40.84067978$, so it is below

$$.6999 \log(.0681) + .0301(40.84067978) < -.65. \quad (\text{F.9})$$

Current stocks below f_0 reduce to f_0 by increasing current resources, while P remains constant. It remains to treat $f_0 \leq K \leq 9/4$ and $K' \geq f_0$.

A decreasing supporting price. Choose a supporting price that cancels all but a resource gap in the residual derivative. Its monotonicity will then let an endpoint check bound the residual. Define

$$\begin{aligned} D_\ell(K) &= F_L(K, \ell_0), & \mathcal{R}(K) &= q_+K + F(K, \ell_0) + (1 - \ell_0)D_\ell(K), \\ m(K) &= \frac{\psi'(K) + dD'_\ell(K)/D_\ell(K)}{\mathcal{R}'(K)}. \end{aligned} \quad (\text{F.10})$$

Put $z_0 = K/(K + \ell_0)$ and write the numerator and denominator as

$$\begin{aligned} N &= a + n_1 + n_2, & J &= q_+ + b_1 + b_2, \\ n_1 &= \frac{5}{2K^{3/2}}, & n_2 &= \frac{5d\ell_0}{2K(K + \ell_0)}, \\ b_1 &= F_K(K, \ell_0), & b_2 &= (1 - \ell_0)D'_\ell(K). \end{aligned}$$

All four terms are positive. Their logarithmic derivatives, multiplied by K , are respectively $-3/2, -1 - z_0, 1/2 - 5z_0/2, 3/2 - 7z_0/2$. Since $z_0 \leq 72/97 < 4/5$, every $n'_i b_j - n_i b'_j$ is negative. Moreover,

$$b_1 \leq \frac{12A_+}{25}\sqrt{5}, \quad b_2 \leq \frac{126A_+}{2401}\sqrt{21}, \quad b_1 + b_2 < 1.$$

The two maxima occur at $K = \ell_0/4$ and $3\ell_0/4$. Using $J' \geq -2(b_1 + b_2)/K$ and $N' \leq -3n_1/(2K)$ gives

$$q_+N' - aJ' < \frac{1}{K} \left(-\frac{10q_+}{9} + \frac{4}{5} \right) < 0.$$

Together with the negative cross terms, this proves $m' < 0$.

Let

$$J_* := q_+ + \frac{12A_+}{25}\frac{9}{4} + \frac{126A_+}{2401}\frac{23}{5} < 1.913.$$

Then $J \leq J_*$ and $m \geq (a + n_1)/J_* > \beta_0 a$. Hence the unique positive number

$$t(K) = \left[\frac{5\beta_0}{2(m(K) - \beta_0 a)} \right]^{2/3} \quad (\text{F.11})$$

satisfies $m = \beta_0 \psi'(t)$.

The full labor choice. For fixed K , the function $J_K(L) = (1 - L)F_L(K, L)$ increases and then decreases: the sign of its derivative is the sign of $K - (3K + 4)L + 2L^2$, a strictly decreasing polynomial on $[0, 1]$. The function $mF(K, L) + d \log(1 - L)$ therefore has at most one interior local maximum; its other possible global maximum is zero at $L = 0$. The bounds just established imply

$$\begin{aligned} mJ_K(K/4) &\geq \frac{48A_+\sqrt{10}}{125J_*} \left(a\sqrt{K} + \frac{5}{2K} \right) (1 - K/4) > .356 > d, \\ mF(K, K/4) + d \log(1 - K/4) &\geq \frac{5A_+}{2J_*} (2/5)^{3/2} + d \log(7/16) > .00016. \end{aligned}$$

The capital factor in the first inequality decreases up to $9/4$; its derivative is negative because $aK^{3/2} < 5$. The auxiliary objective is therefore increasing at $L = K/4$, and the second inequality makes its interior maximum dominate zero labor. This maximum lies above $K/4$, where production is concave because $\text{sgn } F_{LL} = \text{sgn}(K - 4L)$. The production tangent at $\ell_0 > K/4$ therefore bounds production at the maximizer and gives

$$\sup_{0 \leq L < 1} \{mF(K, L) + d \log(1 - L)\} \leq m\{F(K, \ell_0) + (1 - \ell_0)D_\ell\} + d \left(\log \frac{d}{mD_\ell} - 1 \right). \quad (\text{F.12})$$

The auxiliary linear-production maximization may allow $L < 0$; doing so only enlarges its value and gives the displayed formula.

Every feasible successor from $K \leq 9/4$ is below 4, because $q_+(9/4) + A_+(18/13)^{3/2} < q_+(9/4) + 2A_+ < 4$. Thus $P(K') \leq \psi(4) = -19.4 < 0$, which permits replacing β by β_0 in an upper bound. For the remaining successors $K' \geq f_0$, concavity of ψ , the consumption inequality $\eta \log C - mC \leq \eta\{\log(\eta/m) - 1\}$, and (F.12) bound the residual by

$$G_e(K) = e \left(\log \frac{e}{m} - 1 \right) + d \left(\log \frac{d}{mD_\ell} - 1 \right) + \beta_0 \psi(t) - mt + m\mathcal{R} - \psi(K), \quad e \in \{.6999, .7001\}. \quad (\text{F.13})$$

Convexity in e reduces the parameter check to its endpoints. The supporting price m was chosen to cancel all other derivative terms:

$$G'_e = m' \left(\mathcal{R} - t - \frac{e + d}{m} \right). \quad (\text{F.14})$$

Since the supporting price decreases, the resource inequality $\mathcal{R} < t + .9998/m$ on $[f_0, 9/4]$ will make the residual increase with capital. An upper endpoint check will then bound the entire interval.

A scalar resource inequality. Set $\omega = .9998$, $\Delta = \beta_0 n_1$, $c = \beta_0 a$ and $u = (m - c)/\Delta > 0$. Concavity of $u^{2/3}$ at $u = .729$ gives $u^{-2/3} \geq 2.7/(.729 + 2u)$. Since $t = Ku^{-2/3}$, it suffices to show

$$Q := 2.7mK + (.729 + 2u)(\omega - m\mathcal{R}) > 0. \quad (\text{F.15})$$

Use

$$y = \frac{\sqrt{K} - \sqrt{\ell_0}}{\sqrt{K} + \sqrt{\ell_0}}, \quad z = y^2, \quad \mathcal{F}(y) = K\Delta J^2 Q.$$

The required capital interval lies inside $[-.597, .259]$ in y , as follows by substituting these rational endpoints in $K = \ell_0[(1 + y)/(1 - y)]^2$. All factors multiplying Q are positive.

The change of variable reduces the sign check to rational functions and one square root. With $k = \ell_0(1 + y)^2/(1 - y)^2$, define

$$n = a + \frac{64\sqrt{2}}{25} \frac{(1 - y)^3}{(1 + y)^3} + \frac{8d}{5} \frac{(1 - y)^4}{(1 + y)^2(1 + y^2)},$$

$$b = \frac{5A_+\sqrt{2}}{1024} (1 + y)^3 (121 + 42y + 121y^2), \quad j = \frac{3A_+\sqrt{2}}{128} (1 + y)(1 - y)^4 (27 + 14y + 27y^2).$$

Here $\Delta = (64\beta_0\sqrt{2}/25)(1-y)^3/(1+y)^3$. Put $X = .729\Delta - 2c$, $Y = 2.7\Delta k + 2\omega$. Direct multiplication of (F.15) gives $\mathcal{F}(y) = R_0(y) + \sqrt{1+y^2}R_1(y)$, where

$$\begin{aligned} R_0 &= k \left[-2q_+kn^2 + nq_+(Y - Xq_+k) + \omega Xq_+^2 + X \left\{ \frac{\omega j^2}{(1+y^2)^7} - \frac{njb}{(1+y^2)^6} \right\} \right], \\ R_1 &= k \left[\frac{j\{n(Y - Xq_+k) + 2\omega Xq_+\}}{(1+y^2)^4} - \frac{b(2n^2 + nq_+X)}{(1+y^2)^3} \right]. \end{aligned} \quad (\text{F.16})$$

Separating even and odd powers will allow different bounds for positive and negative y . Take the even and odd parts of R_0, R_1 to define rational functions R_e, R_o, T_e, T_o of z :

$$R_0(y) = R_e(z) + yR_o(z), \quad R_1(y) = T_e(z) + yT_o(z), \quad \mathcal{F}(y) = E(z) + yO(z),$$

where $E = R_e + \sqrt{1+z}T_e$ and $O = R_o + \sqrt{1+z}T_o$.

The coefficient table supplies the partial fractions used for both signs of y . Put $v = (1+z)^{-1}$ and $w = (1-z)^{-1}$. The enclosures follow by substitution in (F.16) and the stated bracket for $\sqrt{2}$.

Table 3: Coefficient enclosures for the continuation bound

Term	R_e	R_o	T_e	T_o
1	-8.901852803	0	0	0
v	-49.876865181	25.652861160	38.361669792	-16.708281754
v^2	82.061585969	28.861639666	-69.545570080	.156308197
v^3	-56.796703494	-68.600414368	35.925871018	-4.869578368
v^4	56.192602556	72.862704340	-6.930020547	9.657386263
v^5	-43.267088837	-54.135286642	-6.698101411	4.938002847
v^6	23.263949012	14.522788049	0	0
v^7	-5.439687665	-1.648851181	0	0
w	7.460755290	-7.295975192	6.645977347	-6.329045817
w^2	-4.473345696	.822590228	-.316931531	-.316931531
w^3	4.949908435	3.712431326	0	0
w^4	-2.474954218	-2.474954218	0	0

Each nonzero entry is the center of an enclosure of radius 10^{-9} . Displayed zeros are exact.

Positive y : a one-sided remainder. Let $P_4(y) = \sum_{i=0}^4 a_i y^i$ be the Taylor polynomial of $\mathcal{F}$. The preceding formulas give

i	0	1	2	3	4
Lower bound on a_i	.1411	-1.1927	8.5629	-24.3716	15.2802
Upper bound on a_i	.1412	-1.1926	8.5630	-24.3715	15.2803

We prove $\mathcal{F} - P_4 \geq -11y^4$ for $0 \leq y \leq .259$. Only five coefficients of each of E'' and O'' are needed.

Put $r = .0676$. The partial fractions consist of terms $c_1(1+z)^{-p}$, $c_1(1-z)^{-n}$, and $c_1\sqrt{1+z}(1-z)^{-n}$. For the first form, the second derivative has an alternating series with decreasing terms on $[0, r]$, since $p \leq 7$ and $9r < 1$. Its remainder after degree four lies between $-(p)_7 z^5/5!$ and zero, where $(p)_7 = p(p+1) \cdots (p+6)$. Only positive c_1 can lower this remainder.

The other two forms have nonnegative Taylor coefficients. For the mixed form, write $\sqrt{1+z}(1-z)^{-n} = \sqrt{1-z^2}(1-z)^{-n-1/2}$. If $\sqrt{1-z^2} = 1 - \sum_{j \geq 1} b_j z^{2j}$, then $b_j > 0$ and $\sum b_j = 1$. The

coefficients of $(1-z)^{-n-1/2}$ increase for $n \geq 1$; each coefficient of the product therefore lies between zero and the corresponding coefficient of $(1-z)^{-n-1/2}$. Two differentiations preserve this bound. For each term that lowers the bound, the majorant power is at most four. The ratio of successive omitted terms is at most $11r/6 < 1/8$, so the tail is at most $8/7$ times its first term.

For each of E and O , the sums of positive coefficients in the first form are bounded by 157, 88, 24 for, respectively, $p \leq 5/2$, $5/2 < p \leq 9/2$, and $9/2 < p \leq 6$. The sum of absolute values of negative coefficients in the other two forms is at most 17. Consequently the remainder of each second derivative after degree four is at least $-\mathcal{T}$, where

$$\mathcal{T} = \frac{r^5}{5!} \left[157(5/2)_7 + 88(9/2)_7 + 24(6)_7 + \frac{8}{7} 17(4)_7 \right] < 2.491 < \frac{5}{2}.$$

Differentiating at zero gives the lower coefficient bounds

$$\begin{aligned} E''(z) &\geq 30.56 + 94z - 1446z^2 + 4646z^3 - 21517z^4 - \mathcal{T} > 27, \\ O''(z) &\geq -62.615 + 272z - 2798z^2 + 6177z^3 - 23926z^4 - \mathcal{T} > -66. \end{aligned}$$

For the last inequalities, drop the positive cubic terms and use $z(94 - 1446z) \geq r(94 - 1446r)$ and $z(272 - 2798z) \geq 0$. Integrating the bounds on E'' and O'' twice from $z = 0$, then setting $z = y^2$, gives

$$\mathcal{F}(y) - P_4(y) \geq -[15.2803 - 13.5 + 33y]y^4 \geq -11y^4.$$

Even the weaker bound $-(23/2)y^4$ suffices: it gives

$$\mathcal{F}(y) \geq q_4(y) := .1411 - 1.1927y + 8.5629y^2 - 24.3716y^3 + 3.7802y^4.$$

On $[0, .259]$, $q'_4 \leq -1.1927 + 17.1258y - 69.1985128y^2 < 0$, because the quadratic has negative discriminant. Finally $q_4(.259) = .0001773011962922 > 10^{-4}$.

Negative y : rational monotonicity. For $y \leq 0$, the signs $E > 0$ and $O < 0$ make $\mathcal{F}(y) = E(y^2) + yO(y^2)$ positive. It suffices to prove these signs on $0 \leq z \leq 9/25$. Here $25/34 \leq v \leq 1$, $1 \leq w \leq 25/16$ and $0 \leq s := 1 - v \leq 9/34$. The seven partial fractions of T_o give

$$T_o < .157 + 9.658 + 4.939 - 16.708(25/34) - 4.869(25/34)^3 - 6.329 - .3169 < -6.11.$$

Thus $O \leq R_o + T_o$. Write $R_o + T_o = A(v) + M(w)$. Expansion of the degree six derivative at $v = 1 - s$ gives

$$A'(1-s) > 6.25 - 7.85s^2 - 17.89s^5 - 11.55s^6 > 5.67;$$

the omitted coefficients are positive. Also $M''(w) < 1.012 + 22.275w - 29.699w^2 < 0$ for $w \geq 1$, and $M'(1) < -11.37$. As z increases, v falls and w rises. Hence

$$O(z) \leq (R_o + T_o)(0) < -1.19. \tag{F.17}$$

For the even part, put $H = R_e + T_e$. The identities $(v^k - 1)/z = -\sum_{j=1}^k v^j$ and $(w^k - 1)/z = \sum_{j=1}^k w^j$ give

$$\frac{H(z) - H(0)}{z} = P_7(v) + N_4(w).$$

The degree seven polynomial obeys

$$P_7(1-s) > .36 - 20s + 68s^2 - 90.576s^3 - 39.429s^5 - 5.440s^7 \geq .36 - 20s + 68s^2 - 94s^3 > -2.$$

Its omitted even coefficients are positive. The second inequality uses $s \leq 9/34$; the final cubic decreases and exceeds -1.913 at that endpoint. Moreover,

$$N_4(w) = w\{b_0 + b_1w + b_2w^2(1-w)\}, \quad b_0 > 11.79, \quad b_1 > -2.316, \quad 0 < b_2 < 2.475,$$

so $N_4(w) \geq w\{11.79 - 2.316(25/16) - 2.475(25/16)^2(9/16)\} > 4.7$. Therefore $H(z) > .14 + 2.7z$, including $z = 0$.

Finally write $T_e = C(v) + D(w)$. Its derivatives satisfy

$$\begin{aligned} C'(v) &< 38.362 - 139.09v + 107.778v^2 - 27.720v^3 - 33.490v^4 \\ &\leq 38.362 - 31.312v - 27.720v^3 - 33.490v^4 < 0, \\ D'(w) &> 6.6459 - .634w > 0. \end{aligned}$$

The last bound for C' follows by evaluation at $v = 25/34$, since its displayed upper bound decreases. Thus T_e increases with z and $T_e(z) \geq T_e(0) > -3$. Since $0 \leq \sqrt{1+z} - 1 \leq z/2$,

$$E = H + (\sqrt{1+z} - 1)T_e > .14 + 1.2z > 0.$$

Together with (F.17), this proves $\mathcal{F}(y) = E(y^2) + yO(y^2) > 0$ for $-.6 \leq y \leq 0$. The resource inequality follows on the entire required capital interval.

Since $m' < 0$ and $e+d \geq .9998$, the resource bound gives $G'_e > 0$ in (F.14). The upper endpoint therefore suffices. At $K = 9/4$, the defining formulas and Lemma 11 give

$$G_{.6999}(9/4) < -.00984, \quad G_{.7001}(9/4) < -.00992.$$

Thus the residual for P is below $-.0098$ throughout $[f_0, 9/4]$ when $K' \geq f_0$. Equation (F.9) covers the other successors and current stocks below the floor.

The affine continuation and the join. Let $U(K) = .9K - 23$, $\mu = .9\beta_+$ and $D_a = \beta_+^{-1} - q_+ > 0$. The constant in U is negative, whereas its slope is positive. Accordingly,

$$\begin{aligned} u(C, L) + \beta U(K') - U(K) &\leq \max_{e \in \{.6999, .7001\}} e\{\log(e/\mu) - 1\} + J_a(K, L) + .6923, \\ J_a(K, L) &= \mu F(K, L) - \mu D_a K + d \log(1 - L). \end{aligned}$$

A positive maximum of J_a is interior: the boundary values are nonpositive, large capital has a negative linear cost, and labor tending to one has utility tending to minus infinity. Its first order conditions are $F_K = D_a$ and $\mu F_L(1 - L) = d$. With $x = K/L$ and $\mathcal{D} = D_a^2/(18A_+^2)$, they imply

$$L(x) = \mathcal{D}(1+x)^5/x, \quad x^2 - \mathcal{D}x(1+x)^5 - \frac{d}{\mu D_a} = 0.$$

The coefficients of this degree six polynomial have signs $-, -, +, -, -, -, -$ in increasing order. Descartes' rule and rational substitution place its only two positive roots in $(1, 2)$ and $(3.751, 3.752)$. At the first, $L < L(2) < 1/5$, and the labor condition gives

$$J_a \leq dL \left\{ \frac{2x-1}{3x(1-L)} - 1 \right\} < 0.$$

At the second, Euler's identity gives $J_a = T(x) + d \log(1 - L(x))$, where $T(x) = \mu D_a \mathcal{D}(1+x)^5(2x-1)/3$. Both L and T increase on its bracket, so

$$J_a \leq T(3.752) + d \log(1 - L(3.751)) < .161.$$

The consumption maximum is below $-.8546$. The affine Bellman residual is therefore below $-.8546 + .161 + .6923 = -.0013 < -1/1000$ at every capital stock.

To check transitions across the join, for $K \geq f_0$ write $x = \sqrt{K}$. Then

$$U(K) - \psi(K) = \frac{(x-2)(x^2+2x-5)}{2x}.$$

Below the floor, $\psi(f_0) < -40 < U(K)$. Hence $\Psi \leq U$ everywhere and $\Psi \leq P$ on $[0, 4]$. For a current stock on the curved branch, $K \leq \alpha < 9/4$, every successor is below 4, where $\Psi \leq P$. The inequality for P therefore covers transitions to either branch of Ψ . On the affine branch, use the global affine inequality and $\Psi \leq U$. This proves (F.7).

Since $(1 - \beta)10/301 \leq .001$ throughout the box, (F.7) gives

$$r(e) + \beta \Psi^\sharp(K') - \Psi^\sharp(K) \leq -.001 + (1 - \beta) \frac{10}{301} \leq 0.$$

The terminal condition follows from the resource bound: $K_{t+1} \leq q_+ K_t + 2\sqrt{2}A_+$ and $q_+ < 1$ keep every feasible history from a finite stock bounded. Both Ψ and $\Psi^\sharp$ are bounded on that history's capital interval, including near zero. Their discounted terminal terms vanish, so iteration gives $V(K) \leq \Psi^\sharp(K)$. At zero the value is $-\infty$, and the bound holds directly. $\square$

F.3 Proof of Theorem 9

All decimals below are exact rationals. Write $A_\pm, q_\pm, \beta_\pm, \eta_\pm$ for the endpoints of (7.3). Lemma 12 supplies both Ψ and $\Psi^\sharp$; the latter is given by (7.8) on adverse successors. The finite bounds below use the square root and logarithm rules in Lemma 11, with square root scale 10^{16} and 20 logarithm terms. No value function approximation enters the comparisons.

Feasible witnesses. Three tails of the path in Table 4 supply the regional witnesses' continuations. Let T_M, T_H, T_J denote their discounted welfare starting at stocks .135, .375 and .900, respectively. For each row (k, l, s) , use

$$\underline{u}(k, l, s) = \min_{e \in \{\eta_-, \eta_+\}} \{e \log[q_- k + F_{A_-}(k, l) - s] + (1 - e) \log(1 - l)\}.$$

Every consumption in this expression is positive and every reward floor is negative. Discounting these floors at β_+ , with the last reward divided by $1 - \beta_+$, gives

$$T_M > -33.103, \quad T_H > -27.164, \quad T_J > -23.981. \quad (\text{F.18})$$

Table 4: A feasible expansion path supplies three witness continuations. Read dates 0–8, then dates 9 onward. From date 17 the final action repeats forever. The labeled tails supply the welfare floors in (F.18); the path is not asserted to be optimal.

Dates 0–8					Dates 9 onward			
t	K_t	L_t	K_{t+1}	Tail	t	K_t	L_t	K_{t+1}
0	.100	.68	.135		9	1.360	.80	1.590
1	.135	.71	.187	T_M	10	1.590	.80	1.810
2	.187	.74	.263		11	1.810	.80	2.020
3	.263	.76	.375		12	2.020	.80	2.200
4	.375	.76	.510	T_H	13	2.200	.80	2.370
5	.510	.78	.690		14	2.370	.80	2.530
6	.690	.79	.900		15	2.530	.80	2.680
7	.900	.79	1.130	T_J	16	2.680	.80	2.830
8	1.130	.80	1.360		≥ 17	2.830	.78	2.830

The witnesses use initial actions $(L, K') = (.68, .135)$ on M , $(.72, .375)$ on $[h, .8]$, and $(.78, .9)$ on $[\cdot, \infty)$, followed by T_M, T_H, T_J , respectively. Feasibility at each region's lower endpoint and increasing resources prove feasibility throughout. The staging witness never falls below b and reaches H ; the target witnesses remain in H . If $\underline{T}_X$ denotes the corresponding rounded floor in (F.18), define

$$\underline{W}_X(k) = \underline{u}(k, l_X, s_X) + \beta_+ \underline{T}_X.$$

Use the corresponding target witness on each of the two target regions. The actual witness values exceed these bounds. In particular, $\inf_M W_M \geq W_M(b) > -35.579855 > -35.58$ and $W_H(h) > -28.553250$.

Every adverse successor. For $l_0 \geq K/2$, the tangent to $F_A(K, L)$ at l_0 dominates production on $[0, 1]$. Production is concave above $K/4$; below it, production minus its tangent is convex and nonpositive at both endpoints, since the tangent's intercept is $F_A(K, l_0)(l_0 - K/2)/(K + l_0) \geq 0$. Set

$$t = \partial_L F_{A+}(K, l_0), \quad S = q_+ K + F_{A+}(K, l_0) + (1 - l_0)t.$$

Maximizing utility under this affine resource bound yields

$$r(K, s) \leq E^+(K, s; l_0) := \max_{e \in \{\eta_-, \eta_+\}} \{\log(S - s) + e \log e + (1 - e) \log(1 - e) - (1 - e) \log t\}. \quad (\text{F.19})$$

The endpoint maximum follows from convexity in e . Allowing the auxiliary optimum outside the original labor interval preserves an upper bound.

For $a = b$ or h , the function $E^+(K, s; l_0) + \beta_- \psi(s)$ is concave on $s \in [.05, a]$: only $\log(S - s)$ and $\psi(s)$ depend on s . Its derivative at a is positive if $\beta_- \psi'(a) > 1/(S - a)$. This holds already at $(K, a, l_0) = (b, b, .44)$ and $(h, h, .6)$; increasing S preserves it at larger origins. The upper bound increases up to a on $[\cdot, .05]$. Below .05, the ceiling is constant and current utility decreases with saving, so its largest bound is at zero. Every adverse successor is therefore covered by the larger of

$$E^+(K, a; l_0) + \beta_- \psi(a), \quad E^+(K, 0; l_0) + \beta_- \psi(.05). \quad (\text{F.20})$$

Because the ceiling values are negative, the lower discount endpoint gives the required upper bound.

For each finite origin interval $[k_-, k_+]$ in region X of Table 5, subtract (F.20) evaluated at $K = k_+$ from $\underline{W}_X(k_-)$. Use $(a, l_0) = (b, .44)$ for $X = M$ and $(h, .6)$ for $X = H$. Production monotonicity validates these origin endpoint bounds. The column using Ψ reports these margins, rounded down; each exceeds .01. The column using $\Psi^\sharp$ incorporates the ceiling shift.

Table 5: Certified lower bounds on adverse departure losses. Each row covers every adverse successor and unrestricted continuation, including recovery. The last row follows from (F.21).

Region	Origin interval	Witness advantage	
		Using Ψ	Using $\Psi^\sharp$
M	$[\text{.100}, \text{.102}]$	$> .0125$	$> \mathbf{.0447}$
M	$[\text{.102}, \text{.110}]$	$> .0160$	$> \mathbf{.0482}$
M	$[\text{.110}, \text{.143}]$	$> .0114$	$> \mathbf{.0436}$
M	$[\text{.143}, \text{.280}]$	$> .0342$	$> \mathbf{.0664}$
H	$[\text{.280}, \text{.281}]$	$> .0167$	$> \mathbf{.0489}$
H	$[\text{.281}, \text{.284}]$	$> .0110$	$> \mathbf{.0432}$
H	$[\text{.284}, \text{.290}]$	$> .0150$	$> \mathbf{.0472}$
H	$[\text{.290}, \text{.304}]$	$> .01008$	$> \mathbf{.0423}$
H	$[\text{.304}, \text{.334}]$	$> .0145$	$> \mathbf{.0467}$
H	$[\text{.334}, \text{.402}]$	$> .0108$	$> \mathbf{.0430}$
H	$[\text{.402}, \text{.560}]$	$> .0106$	$> \mathbf{.0428}$
H	$[\text{.560}, \text{.800}]$	$> .1438$	$> \mathbf{.1760}$
H	$[\text{.800}, \infty)$	> 2.2662	$> \mathbf{2.2984}$

Using $\Psi^\sharp$ increases each advantage obtained with Ψ by at least

$$\beta_- \frac{10}{301} = \frac{9699}{301000} > .03222259.$$

All displayed bounds are rounded down. Every adverse margin obtained with $\Psi^\sharp$ exceeds .04222259, so .04 is a common certified margin.

For $K \geq .8$, put $\sigma = .78$, $R = \sigma^{-3/2}$, $Z = qK + F_A(K, \sigma)$, and $Z_0 = q_-(.8) + F_{A_-}(.8, \sigma)$. Full capacity resources are at most RZ . Moreover $Z_0 > .9$, $R(.9) > h$, and $(RZ - s)/(Z - .9)$ decreases in Z for $s \leq h$. Dropping the adverse action's nonpositive leisure utility shows that the third witness exceeds it by at least

$$\mathcal{D}(s) = \beta_- [\underline{T}_J - \Psi(s)] - \eta_+ \log \frac{RZ_0 - s}{Z_0 - .9} + (1 - \eta_-) \log(.22). \quad (\text{F.21})$$

Here $\underline{T}_J > \Psi(s)$ and the log ratio is positive, which justify the parameter endpoints. On $[\text{.05}, h]$, $\mathcal{D}'$ increases and $\mathcal{D}'(h) < 0$, so its minimum is at h ; below .05 its minimum is at zero. The finite checks give $\mathcal{D}(h) > 2.2662$ and $\mathcal{D}(0) > 14.7622$. This covers arbitrarily large current stocks and every future recovery path through the unrestricted ceiling. Replacing Ψ by $\Psi^\sharp$ increases this bound by at least $\beta_- 10/301$ as well.

Calibration and the constant barrier obstruction. Using (F.19) at $(K, s, l_0) = (h, b, .4)$, and the feasible labor choice .4 for the lower bound, gives

$$-.999 < r(h, b) < -.996799.$$

Also $\eta_- \beta_- q_- (h - b) / [q_+ h + F_{A_+}(h, 1) - b] > .2997787$. These inequalities establish (7.6) uniformly. The bounded barrier verification is in the text. Its upper bound is

$$B_{\max} = \frac{-.996}{1 - .9699} - \frac{.2997}{1 - .9699^2} < -38.144176.$$

Together with $W_M > -35.58$, this gives calibration margin > 2.564 . The adverse margins obtained with $\Psi^\sharp$ therefore yield a full margin of .04. The same reward lower bound and $\Psi(b) < -34.271388$ prove the constant barrier obstruction.

Entry dates. Write $\Phi(K) = qK + F_A(K, 1)$. The bounds $\Phi(.105) < .16$ and $\Phi(.16) < .26 < h$ imply $\tau_H \geq 3$ from $[\cdot, .105]$. If a history takes 46 consecutive staying edges, summing 23 copies of (7.6) bounds its discounted rewards over those dates by $(U - \Delta_2/(1 - \beta^2))(1 - \beta^{46})$. Since Ψ increases on M and $\Psi(h) < -27.83$, unrestricted continuation gives total welfare at most

$$B_{\max} + .9701^{46}(-27.83 - B_{\max}) < -35.591511 < -35.58.$$

Such a history is worse than the staging witness. The adverse comparisons exclude downward optimal departures and every subsequent return below h . Theorem 5 therefore gives the asserted entry and retention for every optimum. $\square$

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